On some FK-spaces of statistically convergent sequences

DISSERTATION

SUBMITTED FOR THE AWARD OF THE DEGREE OF

Master of Philosophy

IN

MATHEMATICS

BY

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2012

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Certificate

This is to certify that the dissertation entitled **"On some FK-spaces of statistically convergent sequences"**, has been written by **Mr. Faisal Khan** under my supervision in the department of Mathematics, Aligarh Muslim University, Aligarh in partial fulfillment for the award of the degree of Master of Philosophy in Mathematics. I further certify that the exposition has not been previously submitted to any other University or Institution for the award of any degree.

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Dedicated
to
My Family

ACKNOWLEDGMENT

The accomplishment of this endeavour would not have been feasible without the will of Almighty Allah, for it is his blessing, which provided me enough zeal for the completion of this dissertation in time.

I have immense pleasure to express my gratefulness and indebtness to my supervisor **Prof. Mursaleen**, Department of Mathematics. AMU, Aligarh, for boosting up my panache and prodding me to accomplish this task and continuous guidance and invaluable suggestions instilled me with morale needed to complete the work. The critical comments, he rendered during the discussion have gone a long way in my understanding and presentation of the contents of this dissertation.

I am immensely grateful to the Chairman Prof. Afzal Beg, Department of Mathematics, AMU, Aligarh for providing me all the necessary facilities.

I would like to express my gratitude to all my seniors especially Mr. Musavvir Ali, Mr. Shuja Haider Rizvi, Mr. Abu Zaid Ansari, Mr. Malik Rashid Jamal, Mr, Salahuddin Khan, Mr. Fahad Sikandar, Mr. Phool Miyan Mr. Asif khan and Mr. Ejaz Mustafa for their generous help, suggestions and co-operation.

Its my pleasure to express my deep sense of appreciation to all my friends and colleagous especially Mr. Kamran Khan, Mr. Shahoor Khan, Mr. Vishal Kumar Yadav, Mr. Shoaib Khan, Mr. Izharuddin, Miss. Sana Khan, Miss. Taranum Khan, Miss. Nazia Parveen, Miss Shikha Varshney and Mr. Abdul Nadim Khan who always supported my aspiration with ease and love at varios stages of this work.

I have no words to express my gratitude and thanks to my parents for their limitless sacrifices to enrich my future. They were always with me in good as well as in bad times alike to keep me focussed towards my goal. I would like to express my special thanks to my dearest brothers for their best wishes.

Dated: 02-01-2012

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Preface

The central theme of this dissertation entitled, "On some FK-spaces of statistically convergent sequences", for single as well as double sequences of through which we have established results of various natures, e.g. limit point and cluster point, limit superior and limit inferior, matrix transformations, core theorems, some special matrices etc.

The present dissertation consists of six chapters In Chapter 1, we recall some elementary definitions, notations and background material. Chapter 2 concerns with the study to space of statistically convergent sequences, which was introduced by Fast [11] in 1951, statistically Cauchy sequences and Tauberian theorems.

In Chapter 3, we propose to study the concept of statistical limit points and cluster points, limit superior and limit inferior. There are several well known theorems that are equivalent to the completeness theorems for statistical convergence.

In Chapter 4 and 5 are devoted to study core theorems, strong p- cesàro convergence, inclusion and equivalence theorems and matrix summability results.

In last Chapter 6, we discussed the concepts of statistically convergent and statistically Cauchy double sequences and give the relation between them we also give the relation between statistical convergence and strongly Cesàro summable sequences.

Towards the end of the dissertation, we have given a fairly exhaustive bibliography of the books and publications to which references have been made throughout the dissertation.

CONTENTS

CHAPTER 1

NOTATIONS AND PRELIMINARIES

In this chapter we give some definitions and notations which will be used throughout the dissertation.

1.1. Notations

$\mathbb{N}$:= The set of all natural numbers

$\mathbb{R}$:= The set of all real numbers

$\mathbb{C}$:= The set of all complex numbers

$\lim_k$: $\lim_{k\to\infty}$

$\inf_k$: $\inf_{k\geq 1}$, unless otherwise stated

$\sup_k$: $\sup_{k\geq 1}$, unless otherwise stated

$\sum\limits_k$: summation over $k=1$ to $k=\infty$, unless otherwise stated

$x = (x_k)$ or $\{x_k\}$, the sequence whose k-th term is x_k

$e_k = (0,0,...,0,1,0,0,...)$, the sequence whose k-th component is 1 and others zeros, for all $k \in \mathbb{N}$

$e = (1,1,1,...)$

If g is a positive function of a variable which tend to a limit, then we write

$f = O(g)$: means $|f| < Mg$, where M is constant

$f = o(g)$: means $f/g \to 0$.

1.2. Convergence of real sequences

Definition 1.2.1. A sequence $\{S_n\}$ is said to converge to a real number l (or to have the real number l as its limit) if for each $\epsilon > 0$, there exists a positive integer m (depending on ϵ) such that $| S_n - l |< \epsilon$, for all $n \geq m$.

The same thing expressed in symbols is

$$S_n \to l \text{ as } n \to \infty \text{ or } \lim_{n\to\infty} S_n = l.$$

Theorem 1.2.1. Every convergent sequence is bounded.

Remark 1.2.1. The converse of the above theorem may not be true. For example the sequence $\{S_n\}$, where $S_n = (-1)^n$, $n \in \mathbb{N}$, is bounded but it is not convergent.

Theorem 1.2.2. A sequence cannot converge to more than one limit.

1.3. Limit points of a real sequence

Definition 1.3.1. A real number ξ is said to be a limit point of a sequence $\{S_n\}$. If every neighbourhood of ξ contains an infinite number of members of the sequence.

Thus ξ is a limit point of a sequence if given any positive number ϵ, however small, $S_n \in (\xi - \epsilon,\ \xi + \epsilon)$ for an infinite number of values of n.
i.e.,

$$|S_n - \xi| < \epsilon, \quad \text{for infinitely many values of } n.$$

Example 1.3.1. The sequences $\{S_n\}$, where $S_n = \frac{1}{n}$, $n \in \mathbb{N}$, has 0 as a limit point which is as well as a limit point of the range $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$.

Example 1.3.2. 0 and 2 are the only limit points of the sequence $\{S_n\}$, where $S_n = 1 + (-1)^n$, $n \in \mathbb{N}$. The range set $\{0, 2\}$ has no limit point.

Theorem 1.3.1. (Bolzano-Weierstrass Theorem) Every bounded sequence has a limit point.

Theorem 1.3.2. Every bounded sequence with a unique limit point is convergent.

Remark 1.3.1. The converse of the above theorem is not always true, for there do exists unbounded sequences having a unique limit point.
For example $\{1, 2, 1, 4, 1, 6, ...\}$ has a unique limit point 1, but is not bounded above.

1.4. Limits inferior and superior

Definition 1.4.1. Let $\{a_n\}$ be a sequence of real numbers (not necessarily bounded). We define

$$\liminf_{n\to\infty} a_n = \sup_n \inf\{a_n, a_{n+1}, a_{n+2}, ...\}$$

and

$$\limsup_{n\to\infty} a_n = \inf_n \sup\{a_n, a_{n+1}, a_{n+2}, ...\}$$

as the limit inferior and limit superior respectively of the sequence $\{a_n\}$.

We shall denote limit inferior and limit superior of $\{a_n\}$ by $\underline{\lim}\ a_n$ and $\overline{\lim}\ a_n$ respectively.

Example 1.4.1. If $a_n = (-1)^n,\ n \in \mathbb{N}$, then

$$\overline{\lim}\ a_n = 1, \quad \text{and} \quad \underline{\lim}\ a_n = -1.$$

Theorem 1.4.1. *A bounded sequence $\{a_n\}$ converges to a real number a if and only if*

$$\underline{\lim}\ a_n \ = \ \overline{\lim}\ a_n = a.$$

1.5. Cauchy real sequences

Definition 1.5.1. A sequence $\{S_n\}$ is called a Cauchy sequence or a fundamental sequence if for each $\epsilon > 0$, there exists a positive integer m, such that

$$|S_{n_1} - S_{n_2}| < \epsilon, \qquad \forall\ n_1,\ n_2 \geq m.$$

Remark 1.5.1. In the field of real numbers a sequence is convergent iff it is a Cauchy sequence.

1.6. Sequence spaces

In this section, we recall the definitions and notations for some classical sequence spaces.

$\omega :=\ \{x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}\}$, the space of all sequences, real or complex.

$\ell_\infty :=\ \{x \in \omega : \sup_k |x_k| < \infty\}$, the space of all bounded sequences.

$c :=\ \{x \in \omega : \lim_k x_k = \ell \text{ for some } \ell \in \mathbb{C}\}$, the space of all convergent sequences.

$c_0 :=$ $\{x \in \omega : \lim_k x_k = 0\}$, the space of all null sequences.

$\ell_p := \left\{x = (x_k) \in \omega : \sum_{k=0}^{\infty} |x_k|^p < \infty\right\}$; $(1 \leq p < \infty)$, the space of all absolutely p-summable sequences.

$\ell_1 := \{x = (x_k) \in \omega : \sum\limits_{k=0}^{\infty} |x_k| < \infty\}$, the space of all sequences with associated absolutely convergent series.

$cs := \left\{x = (x_k) \in \omega : \sum_{k=0}^{\infty} x_k \textit{ converges}\right\}$, the space of all sequences with associated convergent series.

$bs := \left\{x = (x_k) \in \omega : \sup_n \left|\sum_{k=0}^{n} x_k\right| < \infty\right\}$, the space of all sequences with associated bounded series.

$bv := \left\{x = (x_k) \in \omega : \sum_{k=0}^{\infty} |x_k - x_{k-1}| < \infty\right\}$; $(x_{-1} = 0)$, the space of all sequences with associated of bounded variation series.

$w_p :=$ $\{x \in \omega : \lim_n \frac{1}{n} \sum\limits_{k=1}^{n} |x_k - \ell|^p$ for some $\ell \in \mathbb{C}\}$, the space of all strongly Cesàro summable sequences, where $0 < p < \infty$.

$C_1 :=$ $\{x \in \omega : \lim_n \frac{1}{n} \sum\limits_{k=1}^{n} x_k = \ell$ for some $\ell \in \mathbb{C}\}$, the space of all Cesàro summable sequences.

Cesàro matrix or C_1-matrix : A matrix $A = (a_{nk})$ such that

$$a_{nk} = \begin{cases} 1/n, & 1 \leq k \leq n \\ 0, & k > n, \end{cases}$$

is called a Cesàro matrix of order 1.

1.7. *FK* and *BK* Spaces

The theory of *FK spaces* is the most powerful and widely used tool in the characterization of matrix mappings between sequence spaces and the most important result was that matrix mappings between *FK spaces* are continuous [40, Theorem 4.2.8].

Definition 1.7.1. A linear topoligical space is a linear space X which has a topology T such that the algebraic operations addition and scalar multiplication are continuous in X.

The continuity of addition means that $f : X \times X \to X$ defined by $f(x, y) = x+y$ is continuous on $X \times X$ and by the continuity of scalar multiplication we mean that $f : \mathbb{C} \times X \to X$ defined by $f(\lambda, x) = \lambda x$ is continuous on $\mathbb{C} \times X$.

Definition 1.7.2. A linear metric space (X, d) is a linear space X with a translation invariant metric d on X such that addition and scalar multiplication are continuous in (X, d).

Example 1.7.1. The space $\ell(p)$ with metric $d(x, y) = \sum_k |x_k - y_k|^{p_k}$ is a linear metric space, where $0 < p_k < 1$ for all $k \in \mathbb{N}$.

Definition 1.7.3. A paranorm is a function $g : X \to \mathbb{R}$ defined on a linear space X such that for all $x, y, z \in X$

(i) $g(x) = 0$ if $x = \theta$

(ii) $g(-x) = g(x)$

(iii) $g(x + y) \leq g(x) + g(y)$

(iv) If $\{\lambda_n\}$ is a sequence of scalars with $\lambda_n \to \lambda_0$ $(n \to \infty)$ and x_n, $a \in X$ with $x_n \to a$ $(n \to \infty)$, in the sense that $g(x_n - a) \to 0$ $(n \to \infty)$, then $\lambda_n x_n \to \lambda_0 a$ $(n \to \infty)$, in the sense that $g(\lambda_n x_n - \lambda_0 a) \to 0$ $(n \to \infty)$.

Definition 1.7.4. (i) A sequence space X with linear topology is called a *K-space* if each of the maps $p_i : X \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for $i = 1, 2, ...$

(ii) A *Frèchet space* is a complete linear metric space.

(iii) *K-space* X is called an *FK space* if X is a complete linear metric space. In other words, we say that X is an *FK-space* if X is a *Frèchet space* with continuous coodinate projection, we mean if $x^{(n)} \to x$ $(n \to \infty)$ in metric of X then $x_k^{(n)} \to x_k$ $(n \to \infty)$ for each $k \in \mathbb{N}$, i.e. for each $k \in \mathbb{N}$, the linear functional $p_k(x) = x_k$ is such that p_k is continuous on X, i.e. X is *K-space*.

Example 1.7.2. Let $A = (a_{nk})_{n,k=1}^{\infty}$ be a triangular matrix, then c_A is an *FK space*, where

$$c_A = \{x = \{x_k\} : Ax \in c\}$$

A normed *FK space* is called a *BK space*, i.e. a *BK space* is a Banach sequence space with continuous coordinates [10, p.345].

The famous example of an FK space which is not a BK space is the space (w, d_w), where

$$d_\omega(x,y) = \sum_{k=0}^{\infty} \frac{1}{2^k}\left(\frac{|x_k - y_k|}{1 + |x_k - y_k|}\right); \quad (x, y \in \omega).$$

On the other hand, the classical sequence spaces are BK spaces with their natural norms. More precisely, the spaces ℓ_∞, c and c_0 are BK spaces with the sup-norm given by $\|x\|_{\ell_\infty} = \sup_k |x_k|$. Also, the space ℓ_p $(1 \le p < \infty)$ is a BK space with the usual ℓ_p-norm defined by $\|x\|_{\ell_p} = (\sum_{k=0}^{\infty} |x_k|^p)^{1/p}$. Further, the spaces bs, and cs are BK spaces with the same norm given by $\|x\|_{bs} = \sup_n |\sum_{k=0}^{n} x_k|$, and bv is a BK space with $\|x\|_{bv} = \sum_{k=0}^{\infty} |x_k - x_{k-1}|$.

A sequence $(b_k)_{k=0}^{\infty}$ in a linear metric space (X, d) is called a *Schauder basis* (or briefly basis) for X if for every $x \in X$ there exists a unique sequence $(\alpha_k)_{k=0}^{\infty}$ of scalars such that $x = \sum_{k=0}^{\infty} \alpha_k b_k$, that is $d(x, \sum_{k=0}^{n} \alpha_k b_k) \to 0$ $(n \to \infty)$. The series $\sum_{k=0}^{\infty} \alpha_k b_k$ which has the sum x is called the expansion of x and (α_k) is called the sequence of coefficients of x with respect to the basis (b_k) [31, Definition 8.21].

1.8. Matrix transformations

If A is an infinite matrix with complex entries a_{nk} $(n, k \in \mathbb{N})$, then we may write $A = (a_{nk})$ instead of $A = (a_{nk})_{n,k=0}^{\infty}$. Also, we write A_n for the sequence in the n^{th} row of A, i.e. $A_n = (a_{nk})_{k=0}^{\infty}$ for every $n \in \mathbb{N}$. In addition, if $x = (x_k) \in \omega$ then we define the A-*transform* of x as the sequence $Ax = (A_n(x))_{n=0}^{\infty}$, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k, \quad (n \in \mathbb{N}) \tag{1.8.1}$$

provided the series on the right converges for each $n \in \mathbb{N}$. Further, the sequence x is said to be A-*summable* to the complex number ℓ if Ax converges to ℓ which is called the A-*limit* of x.

Let X and Y be subsets of ω and A an infinite matrix. Then, we say that A defines a *matrix mapping* from X into Y if Ax exists and is in Y for every $x \in X$. By (X, Y), we denote the class of all infinite matrices that map X into Y.

An infinite matrix A is said to be *regular* (or *Toeplitz*) if $A \in (c, c)$ and every sequence $x \in c$ is A-summable to the same ordinary limit of x. The well known Silverman-Toeplitz conditions for A to be regular are:

(i) $\|A\| = \sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$

(ii) $\lim_{n\to\infty} a_{nk} = 0$ for each $k \in \mathbb{N}$

(iii) $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{\infty} a_{nk} = 1$.

We refer the reader to [10, 24, 39] for the characterization of matrix transformations between some sequence spaces. For example, we have the following results (see [39, pp.2-9]) which may be needed in the sequel.

Lemma 1.8.1. *We have* $(\ell_\infty, \ell_\infty) = (c, \ell_\infty) = (c_0, \ell_\infty)$. *Further* $A \in (\ell_\infty, \ell_\infty)$ *if and only if*

$$\sup_n \left(\sum_{k=0}^{\infty} |a_{nk}| \right) < \infty. \tag{1.8.2}$$

Lemma 1.8.2. *We have the following:*

(a) $A \in (c, c)$ *if and only if* (1.8.2) *holds and*

$$\lim_{n\to\infty} a_{nk} \text{ exists for every } k \in \mathbb{N}, \tag{1.8.3}$$

$$\lim_{n\to\infty} \left(\sum_{k=0}^{\infty} a_{nk} \right) \text{ exists.}$$

(b) $A \in (c, c_0)$ *if and only if* (1.8.2) *holds and*

$$\lim_{n\to\infty} a_{nk} = 0 \text{ for all } k \in \mathbb{N}, \tag{1.8.4}$$

$$\lim_{n\to\infty} \left(\sum_{k=0}^{\infty} a_{nk} \right) = 0.$$

Lemma 1.8.3. *We have* $A \in (\ell_\infty, c)$ *if and only if* (1.8.2) *and* (1.8.3) *hold and*

$$\lim_{n\to\infty} \left(\sum_{k=0}^{\infty} |a_{nk} - \alpha_k| \right) = 0,$$

where $\alpha_k = \lim_{n\to\infty} a_{nk}$ *for all* $k \in \mathbb{N}$.

Lemma 1.8.4. *Let* $1 < p < \infty$ *and* $q = p/(p-1)$, *then we have*

(a) $A \in (\ell_p, c_0)$ *if and only if* (1.8.4) *holds and*

$$\sup_n \left(\sum_{k=0}^{\infty} |a_{nk}|^q \right) < \infty.$$

(b) $A \in (\ell_1, c_0)$ *if and only if* (1.8.4) *holds and*

$$\sup_{n,k} |a_{nk}| < \infty.$$

(c) $A \in (\ell_\infty, c_0)$ *if and only if*

$$\lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|a_{nk}|\right)=0.$$

(d) $A \in (c_0, c_0)$ *if and only if* (1.8.2) *and* (1.8.4) *hold.*

Lemma 1.8.5. *Let $1 \le p < \infty$, then we have $A \in (\ell_1, \ell_p)$ if and only if*

$$\sup_k\left(\sum_{n=0}^{\infty}|a_{nk}|^p\right)<\infty.$$

Further, let $\mathcal{F}$ be the collection of all nonempty and finite subsets of $\mathbb{N} = \{0, 1, 2, \ldots\}$ throughout, then we have the following results:

Lemma 1.8.6. *Let $1 < p < \infty$ and $q = p/(p-1)$, then we have $A \in (\ell_p, \ell_1)$ if and only if*

$$\sup_{N\in\mathcal{F}}\left(\sum_{k=0}^{\infty}\Big|\sum_{n\in N}a_{nk}\Big|^q\right)<\infty.$$

1.9. Core of a sequence

Definition 1.9.1. The *core* or *K-core* of a real number sequence $x = (x_k)$ is defined to be the closed interval $[\liminf x, \ \limsup x]$.

Definition 1.9.2. If x is a complex number sequence then its core is defined as

$$K\text{-}core\{x\} = \bigcap_{n=1}^{\infty} C_n(x),$$

where $C_n(x)$ is closed convex hull of $(x_k)_{k\ge n}$.

In [37] it is shown that for every bounded x

$$K\text{-}core\{x\} = \bigcap_{z\in\mathbb{C}} B_x^*(Z),$$

where

$$B_x^*(Z) = \left\{w \in \mathbb{C} : |w - z| \le \limsup_k |x_k - z|\right\}.$$

The well-known Knopp core theorem states as follows (see Knopp [21], Maddox [26]).

Theorem 1.9.1. (Knopp's Core Theorem) In order that $L(Ax) \leq L(x)$ for every bounded sequence $x = (x_k)$, it is necessary and sufficient that $A = (a_{nk})$ should be regular and $\lim_n \sum_{k=0}^{\infty} \mid a_{nk} \mid = 1$, where $L(x) = \limsup x$.

1.10. Special matrices

Before proceeding further note that we may replace a_{nk} by $a_k(t)$ in most of our proofs and let $t \to \infty$ continuously. Thus, for example if

$$a_k(t) = e^{-k}t^k/k! \quad (k = 0, 1, 2, ..., \ t > 0),$$

then $a_k(t) \to 0 \quad (t \to \infty, k\text{fixed})$ and

$$\sum_{k=0}^{\infty} |a_k(t)| = \sum_{k=0}^{\infty} a_k(t) = 1,$$

whence $\sup_t \sum |a_k(t)| < \infty$. We shall still call $A = (a_k(t))$ a Toeplitz matrix.

Definition 1.10.1. (Borel matrix) The Toeplitz matrix defined by

$$a_k(t) = e^{-t}t^k/k! \quad (k = 0, 1, 2, ..., \ t > 0).$$

Definition 1.10.2. (Cesáro matrix of oder 1) The Toeplitz matrix defined by

$$a_{nk} = 1/(n+1) \quad (0 \leq k \leq n), \quad a_{nk} = 0 \quad (k > n).$$

(That $x_k \to x$ implies $\frac{1}{n+1} \sum_{k=0}^{n} x_k \to x$ was known to Cauchy.)

Definition 1.10.3. (Cesáro matrix of oder r) For each $r > -1$ the (C, r) matrix is defined by

$$a_{nk} = A_{n-k}^{r-1}/A_n^r \quad (0 \leq k \leq n), \quad a_{nk} = 0 \quad (k > n),$$

where $A_n^r = (r+1)(r+2)...(r+n)/n!$ for $n \geq 1$, $A_0^r = 1$.

Each matrix (C, r) is called Cesáro matrix of order r. If $r \geq 0$, then Cesáro matrix is a Toeplitz but if $-1 < r < 0$, then (C, r) is not Toeplitz.

Definition 1.10.4. (Euler-Knopp matrix) Define

$$a_{nk} = {}^nC_k r^k (1-r)^{n-k} \quad (0 \leq k \leq n), \quad a_{nk} = 0 \quad (k > n,)$$

then A is Toeplitz when $0 < r < 1$. We denote the Euler-Knopp matrix of order r by (E, r).

Definition 1.10.5. (Nörlund matrix) Let $p_0 > 0$, $p_n \geq 0 (n \geq 1)$, $P_n = p_0 + ... + p_n$. Define $a_{nk} = p_{n-k}/p_n$ $(0 \leq k \leq n,)$ $a_{nk} = 0$ $(k > n)$. The matrix A defines the Nörlund mean. A is Toeplitz if and only if $p_n/P_n \to 0$ $(n \to \infty)$.

CHAPTER 2

STATISTICAL CONVERGENCE

2.1. Introduction

In [11] in 1951 H. Fast introduced an extention of the usual concept of sequential limit which he called statistical convergence. In 1953, this concept arises as an example of "convergence in density" as introduced by Buck [4]. In [38] I. J. Schoenberg gave some basic properties of statistical convergence and also studied the concept as a summability method. Zygmund [42] established a relation between it and strong summability. The concept of convergence in "density" is generated by matrix and nonmatrix summability methods, has been explored by Freedman and Sember ([12], [13], [14]). In most convergence theories it is desirable to have a criterion that can be used to verify convergence without using the value of the limit. For this purpose we introduce the analogue of the Cauchy convergence criterion. In the last of this chapter statistical convergence is studied as a summability method; the strength of this method is compared with that of general matrix methods and Tauberian theorems are proved.

2.2. Statistically convergent and statistically Cauchy sequences

Definition 2.2.1. Let $K \subseteq \mathbb{N}$. Then the *natural density* of K is defined by

$$\delta(K) = \lim_{n} \frac{1}{n} |\{k \leq n \ : \ k \in K\}|,$$

where $|\{k \leq n : k \in K\}|$ denotes the number of elements of K not exceeding n.

For example, the set of even integers has natural density $\frac{1}{2}$ and set of primes has natural density zero.

Definition 2.2.2. The number sequence x is said to be *statistically convergent* to the number L provided that for each $\epsilon > 0$,

$$\delta(K) = \lim_{n} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| = 0,$$

i.e.,

$$|x_k - L| \ < \ \epsilon \quad a.\ a.\ k. \tag{2.2.1}$$

In this case we write

$$st\text{-}\lim x_k = L.$$

By the symbol st we denote the set of all statistically convergent sequences and by st_0 the set of all statistically null sequences.

Note that every convergence sequence is statistically convergent to the same number, so that statistical convergent is a natural generalization of the usual convergence of sequences.

The sequence which converges statistically need not be convergent and also need not be bounded.

Example 2.2.1. Let $x = (x_k)$ be defined by

$$x_k = \begin{cases} k, & \text{if } k \text{ is a square} \\ 0, & \text{otherwise.} \end{cases}$$

Then $|\{k \leq n : x_k \neq 0\}| \leq \sqrt{n}$. Therefore, $st\text{-}\lim x_k = 0$. Note that we could have assigned any values whatsover to x_k when k is a square, and we could still have $st\text{-}\lim x_k = 0$. But x is neither convergent nor bounded.

It is clear that if the inequality in (2.1.1) holds for all but finitely many k, then $\lim x_k = L$. It follows that $\lim x_k = L$ implies $st\text{-}\lim x_k = L$ so statistical convergence may be considered as a regular summability method. This was observed by Schoenberg [38] along with the fact that the statistical limit is a linear functional on some sequence space. Salat [36] proved that the set of bounded statistically convergent (real) sequences is a closed subspace of the space of bounded sequences.

Definition 2.2.3. The number sequence x is said to be *statistically Cauchy sequence* provided that for every $\epsilon > 0$ there exists a number $N(= N(\epsilon))$ such that

$$|x_k - x_N| < \epsilon \;\; a.\ a.\ k, \tag{2.2.2}$$

i.e.,

$$\lim_n \frac{1}{n}|\{k \leq n : |x_n - x_N| \geq \epsilon\}| = 0.$$

In order to prove the equivalence of Definitions 2.2.2 and 2.2.3 we shall find it helpful to use a third (equivalent). This property states that for almost all k, the values x_k coincide with those of a convergent sequence.

Theorem 2.2.1. The following statements are equivalent:

(i) x is a statistically convergent sequence

(ii) x is a statistically Cauchy sequence

(iii) x is a sequence for which there is a convergent sequence y such that $x_k = y_k \;\; a.\ a.\ k.$

Proof. To prove that (i) implies (ii) we use an adaptation of the familiar proof that a convergent sequence is a Cauchy sequence. Suppose $st\text{-}\lim x_k = L$ and $\epsilon > 0$, then $|x_k - L| < \epsilon/2 \;\; a.\ a.\ k$, and if N is choosen so that $|x_N - L| < \epsilon/2$ then we have

$$\begin{aligned} |x_k - x_N| &< |x_k - L| \; + \; |x_N - L| \\ &< \epsilon/2 + \epsilon/2 \;\; a.\ a.\ k. \end{aligned}$$

Hence, x is a statistically Cauchy sequence.

Next, assume (ii) is true and choose N so that the interval $I = [x_N - 1, x_N + 1]$ contains $x_k \;\; a.\ a.\ k$. Also apply (ii) to choose M so that $I' = [x_M - 1/2, x_M + 1/2]$ contains $x_k \;\; a.\ a.\ k$. We assert that

$$I_1 = I \cap I' \;\text{ contains } x_k \;\; a.\ a.\ k.$$

for,

$$\{k \le n : x_k \notin I \cap I'\} = \{k \le n : x_k \notin I\} \cup \{k \le n : x_k \notin I'\},$$

therefore,

$$\begin{aligned} &\lim_n \tfrac{1}{n}|\{k \le n : x_k \notin I \cap I'\}| \\ &\qquad \le \lim_n \tfrac{1}{n}|\{k \le n : x_k \notin I\}| \; + \; \lim_n \tfrac{1}{n}|\{k \le n : x_k \notin I'\}| = 0. \end{aligned}$$

Therefore I_1 is a closed interval of length less than or equal to 1 contains $x_k \;\; a.a.\ k$. Now we proceed by choosing $N(2)$ so that $I'' = [x_{N(2)} - 1/4, x_{N(2)} + 1/4]$ contains $x_k \;\; a.\ a.\ k$ and by preceding argument $I_2 = I_1 \cap I''$, I_2 contains $x_k \;\; a.\ a.\ k$ and I_2 has length less than or equal to $1/2$. Continuing inductively we construct a sequence $\{I_m\}_{m=1}^{\infty}$ of closed intervals such that for each m, $I_m \supseteq I_{m+1}$, the length of I_m is not greater than 2^{1-m} and $x_k \in I_m$ *a. a.* k. By the Nested Intervals Theorem, there is a number λ equal to $\cap_{m=1}^{\infty} I_m$. Using the fact that $x_k \in I_m$ *a. a.* k we choose an increasing positve integers sequence $\{T_m\}_{m=1}^{\infty}$ such that

$$\frac{1}{n}|\{k \leq n : x_k \notin I_m\}| < \frac{1}{m}, \qquad \text{if } n > T_m. \tag{2.2.3}$$

Now define a subsequence z of x consisting of all terms x_k such that $k > T_1$ and

$$\text{if } T_m < k \leq T_{m+1} \text{ then } x_k \notin I_m.$$

Next define the sequence y by

$$y_k = \begin{cases} \lambda, & \text{if } x_k \text{ is a term of } z \\ x_k, & \text{otherwise.} \end{cases}$$

Then $\lim y_k = \lambda$; for, if $\epsilon > 1/m > 0$ and $k > T_m$ then either x_k is a term of z, which means $y_k = \lambda$ or $y_k = x_k \in I_m$ and $|y_k - \lambda| \leq$ length of $I_m \leq 2^{1-m}$. We also assert that $x_k = y_k$ *a.a.* k. To verify this we observe that if $T_m < n < T_{m+1}$ then

$$\{k \leq n : y_k \neq x_k\} \subseteq \{k \leq n : x_k \notin I_m\},$$

so by (2.2.3)

$$\frac{1}{n}|\{k \leq n : y_k \neq x_k\}| \ \leq \ |\{k \leq n : x_k \notin I_m\}| < \frac{1}{m}.$$

Hence, the limit as $n \to \infty$ is 0 and $x_k = y_k$ *a. a.* k. Therefore (ii) implies (iii).

Finally, assume that (iii) holds, say $x_k = y_k$ *a. a.* k and $\lim y_k = L$. Suppose $\epsilon > 0$. Then for each n,

$$\{k \leq n : |x_k - L| \geq \epsilon\} \subseteq \{k \leq n : y_k \neq x_k\} \cup \{k \leq n : |y_k - L| > \epsilon\},$$

since $\lim y_k = L$, the latter set contains a fixed number of integers, say $\ell = \ell(\epsilon)$. Therefore

$$\begin{aligned} &\lim_n \tfrac{1}{n}|\{k \leq n : |x_k - L| \geq \epsilon\}| \\ &\leq \lim_n \tfrac{1}{n}|\{k \leq n : y_k \neq x_k\}| \ + \ \lim_n \tfrac{\ell}{n} \\ &= 0. \end{aligned}$$

because $x_k = y_k$ *a. a.* k. Hence, $|x_k - L| < \epsilon$ *a. a.* k, so (i) holds. This completes the proof of the theorem.

As an immediate consequence of Theorem 2.2.1 we have the following result:

Corollary 2.2.1. If x is a sequence such that st-$\lim x_k = L$, then x has a subsequence y such that $\lim y_k = L$.

In [38, Lemma 4] Schoenberg proved that the Cesàro mean of order 1 sums every bounded statistically convergent sequence. This raises the question of whether the C_1 *method includes the statistical convergence method irrespective of boundedness.* The answer is negative, a fortiari, as we shall see in the next theorem. But first we give a useful lemma.

Lemma 2.2.1. If t is a number sequence such that $t_k \neq 0$ for infinitely many k, then there is a sequence x such that $x_k = 0$ *a. a.* k and $\sum_{k=1}^{\infty} t_k x_k = \infty$.

Proof. Choose an increasing sequence of positive integers $\{m(k)\}_{k=1}^{\infty}$ such that for each k,

$$m(k) > k^2 \quad \text{and} \quad t_{m(k)} \neq 0.$$

Define x by $x_{m(k)} = 1/t_{m(k)}$ and $x_k = 0$ otherwise. Then $x_k = 0$ *a. a.* k and $\sum_{k=1}^{\infty} t_k x_k = \sum_{k=1}^{\infty} t_{m(k)} x_{m(k)} = \infty$.

Theorem 2.2.2. No matrix summability method can include the method of statistical convergence.

Proof. The preceding Lemma 2.2.1 shows that in order for a matrix to include statistical convergence it would have to be row-finite. Let A be an arbitrary row-finite matrix and choose a nonzero entry, say $a_{n(1),k'(1)} \neq 0$. Then choose $k(1) \geq k'(1)$ so that

$$a_{n(1),k(1)} \neq 0 \quad \text{and} \quad a_{n(1),k} = 0 \quad \text{if } k > k(1).$$

Now select increasing sequences of row and column indices such that for each m,

$$a_{n(m),k(m)} \neq 0, \quad k(m) \geq m^2 \quad \text{and} \quad a_{n(m),k} = 0 \quad \text{if } k > k(m).$$

Define the sequence x as follows:

$$x_{k(1)} = \frac{1}{a_{n(1),k(1)}}, \quad \ldots.$$

$$x_{k(m)} = \frac{1}{a_{n(m),k(m)}} \left[m - \sum_{i=1}^{m-1} a_{n(m),k(i)} x_{k(i)}\right], \quad \ldots ,$$

and $x_k = 0$, otherwise. Then x is not A-summable because $(Ax)_{n(m)} = m$; also, $k(m) \geq m^2$ implies that $|\{k \leq n : x_k \neq 0\}| \leq \sqrt{n}$, so $x_k = 0$ *a. a. k*. Thus $st\text{-}\lim x_k = 0$, we conclude that A does not include statistical convergence.

Remark 2.2.1. By definition, the method of statistical convergence cannot sum any periodic sequence such as $\{(-1)^k\}$. Therefore statistical convergence does not include most of the classical summability methods. When combined with Theorem 2.2.2 this suggests that perhaps statistical convergence cannot be compared to any nontrivial matrix method. The following example shows that is not the case.

Example 2.2.2. Define the matrix A by

$$a_{nk} = \begin{cases} 1, & \text{if } k = n \text{ and } n \text{ is not a square} \\ 1/2, & \text{if } n = m^2 \text{ and if } \ k = n \text{ or } k = (m-1)^2 \\ 0, & \text{otherwise.} \end{cases}$$

Then for any sequence x we have

$$(Ax)_n = \begin{cases} x_1/2, & \text{if } n = 1 \\ \{x_{(m-1)^2} + x_{m^2}\}/2, & \text{if } n = m^2 \ \text{ for } m = 2, 3, \ \ldots. \\ x_n, & \text{if } n \text{ is not a square.} \end{cases}$$

Thus A is obviously a regular triangle. To see that A is included by statistical convergence suppose $\lim_n (Ax)_n = L$. Then $\lim\limits_{n \neq m} x_n = L$ and obviously $|\{k \leq n : (Ax)_n \neq x_n\}| \leq \sqrt{n}$, so by Theorem 2.2.1, $st\text{-}\lim x_k = L$. To see that A is not equivalent to ordinary convergence consider the sequence x given by

$$x_k = \begin{cases} (-1)^m, & \text{if } k = m^2 \text{ for } m = 1, 2, \ \ldots \\ 0, & \text{if } k \text{ is not a square.} \end{cases}$$

Then $(Ax)_n = 0$ for $n > 1$, but x is nonconvergent.

2.3. Tauberian theorems

The remainder of this chapter is concerned with Tauberian theorems. We shall use the notation Δx for the sequence of forward differences; $\Delta x_k = x_k - x_{k+1}$.

We know that every convergent sequence is also statistically convergent but converse need not be true, e.g. see example 2.2.1. Our next theorem presents a condition (Tauberian) under which the statistical convergence implies convergence.

Theorem 2.3.1. If x is a sequence such that $st\text{-}\lim x_k = L$ and $\Delta x_k = O(1/k)$, then $\lim x_k = L$.

Proof. Assume that $st\text{-}\lim x_k = L$ and using Theorem 2.2.1, choose a sequence y such that $\lim y_k = L$ and $x_k = y_k$ *a.a.* k. For each k write $k - m(k) + p(k)$, where $m(k) = \max\{i \leq k : x_i = y_k\}$; if the set $\{i \leq k : x_i = y_k\}$ is empty, take $m(k) = -1$. (This can occur for at most a finite number of k.) We assert that

$$\lim_k \frac{p(k)}{m(k)} = 0. \tag{2.3.1.}$$

For, if $p(k)/m(k) > \epsilon > 0$, then

$$\begin{aligned} \frac{1}{k}|\{i \leq k : x_i \neq y_i\}| &\leq \frac{1}{m(k)+p(k)}p(k) \\ &\leq \frac{p(k)}{p(k)/\epsilon + p(k)} \\ &= \frac{\epsilon}{1+\epsilon}, \end{aligned}$$

so if $p(k)/m(k) \geq \epsilon$ for infinitely many k, we would contradict $x_k = y_k$ *a.a.* k. Thus (2.3.1) holds. Now consider the difference between $y_{m(k)}$ and x_k. Since $\Delta x_k = O(1/k)$ there is a constant B such that $|\Delta x_k| \leq B/k$ for all k. Therefore

$$\begin{aligned} |y_{m(k)} - x_k| &= |x_{m(k)} - x_{m(k)+p(k)}| \\ &\leq \sum_{i=m(k)}^{m(k)+p(k)-1} |\Delta x_i| \\ &\leq p(k)B/m(k). \end{aligned}$$

By (2.3.1) the last expression tends to 0 as $k \to \infty$, and since $\lim y_k = L$, we conclude that $\lim x_k = L$. This completes the proof of the theorem.

The next theorem shows that the term $O(1/k)$ in Theorem 2.3.1 gives the "best possible" Tauberian condition of order type for statistical convergence.

Theorem 2.3.2. If $\{r_k\}$ is a decreasing positive number sequence such that $\{kr_k\}$ is unbounded, i.e. $r_k \neq O(1/k)$, then there exists a sequence x such that st-$\lim x_k = 0$ and $\Delta x_k = O(r_k)$, but x is not convergent.

Proof. Let $\{r_k\}$ be as given above; we shall construct a nonconvergent sequence x satisfying $x_k = 0 \quad a.\ a.\ k$ by separating its blocks of zero terms by blocks of terms that increase from 0 to 1 then decrease back to 0 with increments given by $\mid \Delta x_k \mid = r_k$. The q-th nonzero block would be

$$x_{n(q)} = 0 < x_{n(q)+1} = r_{n(q)} < \ \dots\ < x_{t(q)} \geq 1,$$

$$x_{t(q)} > \ \dots\ > x_{n(q)} > 0 = x_{n(q)+1}.$$

Since $r_{n(q)}$ is the smallest increment in this block we need at most $2[1/r_{n(q)}]$ terms in the block. These nonzero blocks are located by choosing $n(q)$ as follows:
using the hypothesis that $\{kr_k\}$ is unbounded, choose $n(q) > n(q-1)$ so that

$$n(q)r_{n(q)} > 2q^2.$$

It remains to show that st-$\lim x_k = 0$, or in this case, $x_k = 0 \quad a.\ a.\ k$.
Let $A(n) = |\{k \leq n : x_k \neq 0\}|$. It is easy to see that $A(n)/n$ increases when x_n is in a nonzero block and decreases when x_n is in a zero block. Thus, to show that $\lim A(n)/n = 0$, it suffices to show that $\lim_q A(n(q))/n(q) = 0$. To accomplish this we observe that,

$$\begin{aligned}
\frac{A(n(q))}{n(q)} &\leq \frac{1}{n(q)} \sum_{i=1}^{q} [\text{length of } i\text{-}th \text{ nonzero block}] \\
&\leq \frac{1}{n(q)} \sum_{i=1}^{q} [2/r_{n(i)}] \\
&= \frac{2q}{n(q)r_{n(q)}} \\
&< \frac{1}{q}
\end{aligned}$$

Hence $x_k = 0 \quad a.a.\ k$ and $\Delta x_k = O(r_k)$, but x is nonconvergent.
This completes the proof of the theorem.

The final result of this chapter is a Tauberian theorem that uses a "gap condition" instead of the order condition as in Theorem 2.3.1. We say that x is a gap sequence if $\Delta x_k = 0$ except for certain indices k which occur at wide intervals or gaps.

Theorem 2.3.3. If $\{k(i)\}_{i=1}^{\infty}$ be an increasing sequence of positive integers such that $\liminf_i \frac{k(i+1)}{k(i)} > 1$, and let x be a corresponding gap sequence: $\Delta x_k = 0$ if $k \neq k(i)$ for $i = 1, 2, \ldots.$, if $st\text{-}\lim x_k = L$, then $\lim x_k = L$.

Proof. If $\liminf_i \frac{k(i+1)}{k(i)} = 1 + 2\delta > 1$, then for i sufficiently large we have

$$\frac{k(i+1)}{k(i)} > 1 + \delta > 1, \tag{2.3.2}$$

or

$$k(i+1) - k(i) > \delta k(i).$$

This mean that the number of terms in the $(i+1)-$st block (throughout which x_k is constant) is greater than $\delta k(i)$. Now suppose $\lim x_k \neq L$ and choose $\epsilon > 0$ so that for arbitrarily large k, $|x_k - L| \geq \epsilon$. Thus if such a k is chosen from the $(i+1)$-*st* block, where i is large enough to ensure that (2.3.2) holds, we have

$$\begin{aligned}\frac{1}{k(i+1)}|\{k \leq k(i+1) : |x_k - L| \geq \epsilon\}| &> \frac{k(i+1) - k(i)}{k(i+1)} \\ &> \frac{\delta}{1+\delta}.\end{aligned}$$

Hence, $(1/n)|\{k \leq n : |x_k - L| \geq \epsilon\}|$ does not tend to zero, so st-$\lim x_k \neq L$.

This completes the proof of the theorem.

CHAPTER 3

STATISTICAL ANALOGUES OF COMPLETENESS PROPERTIES

3.1. Introduction

In the present chapter we return to the view of statistical convergence as a sequential limit concept and we extend this concept in a natural way to define a statistical analogue of the set of limit points or cluster points of a number sequence. In Section 2 we give the basic properties of statistical limit points and cluster points. This section develops the similarities and differences between these points and ordinary limit points. Section 3 presents statistical analogues of some of the well-known completeness properties of the real numbers.

The purpose of this chapter is to present natural definitions of the concepts of statistical limit superior and inferior and to develop some statistical analogues of properties of the ordinary limit superior and inferior. The latter results include statistical analogues of Knopp's Core Theorem [21] and R.C. Buck's Theorem [4] on Cesáro summability of a sequence to its limit superior.

The zero density property is described succinctly as "$x_k = y_k$ for almost all k". Sets of density zero play an important role, so we introduce some convenient terminology and notation for working with them. If x is a sequence we write $\{x_k : k \in \mathbb{N}\}$ to denote the range of x. If $\{x_{k(j)}\}$ is a subsequence of x and $K = \{k(j) : j \in \mathbb{N}\}$, then we abbreviate $\{x_{k(j)}\}$ by $\{x\}_K$. In case $\delta(K) = 0$, $\{x\}_K$ is called a subsequence of density zero, or a thin subsequence. On the other hand, $\{x\}_K$ is a nonthin subsequence of x if K does not have density zero. It should be noted that $\{x\}_K$ is a nonthin subsequence of x if either $\delta(K)$ is a positive number or K fails to have natural density.

3.2. Statistical limit points and cluster points

The number L is an ordinary limit point of a sequence x if there is a subsequence of x that converges to L; therefore we define a statistical limit point by considering the density of such a subsequence.

Definition 3.2.1. The number λ is a *statistical limit point* of the number sequence x provided that there is a nonthin subsequence of x that converges to λ.

Notation. For any number sequence x, let Λ_x denote the set of statistical limit points of x and L_x denote the set of ordinary limit points of x.

Example 3.2.1. Let $x_k = 1$ if k is a square and $x_k = 0$ otherwise; then $L_x = \{0, 1\}$ and $\Lambda_x = \{0\}$.

It is clear that $\Lambda_x \subseteq L_x$ for any sequence x. To show that Λ_x and L_x can be very different, we give a sequence x for which $\Lambda_x = \phi$ while $L_x = \mathbb{R}$, the set of real numbers.

Example 3.2.2. Let $\{r_k\}_{k=1}^{\infty}$ be a sequence whose range is the set of all rational numbers and define

$$x_k = \begin{cases} r_n, & \text{if } k = n^2 \text{ for } n = 1, 2, 3, ... \\ k, & \text{otherwise.} \end{cases}$$

Since the set of squares has density zero, it follows that $\Lambda_x = \phi$, while the fact that $\{r_k : k \in \mathbb{N}\}$ is dense in $\mathbb{R}$ implies that $L_x = \mathbb{R}$.

A limit point L of a sequence x can be characterized by the statement "every open interval centered at L contains infinitely many terms of x". To form a statistical analogue of this criterion we require the open interval to contain a nonthin subsequence, but we must avoid calling the center of the interval a statistical limit point for reasons that will be apparent shortly.

Definition 3.2.2. The number γ is a *statistical cluster point* of the number sequence x provided that for every $\epsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - \gamma| < \epsilon\}$ does not have density zero.

For a given sequence x, we let Γ_x denote the set of all statistical cluster points of x. It is clear that $\Gamma_x \subseteq L_x$ for every sequence x. The inclusion relationship between Γ_x and Λ_x is a bit more subtle.

Proposition 3.2.1. For any number sequence x, $\Lambda_x \subseteq \Gamma_x$.

Proof. Suppose $\lambda \in \Lambda_x$, say $\lim_j x_{k(j)} = \lambda$, and

$$\limsup \frac{1}{n} |\{k(j) \leq n\}| = d > 0.$$

For each $\epsilon > 0$, $\{j : |x_{k(j)} - \lambda| \leq \epsilon\}$ is a finite set, so

$$\{k \in \mathbb{N} : |x_k - \lambda| < \epsilon\} \supseteq \{k(j) : j \in \mathbb{N}\} \sim \{\text{finite set}\}.$$

Therefore,

$$\frac{1}{n}|\{k \le n : |x_k - \lambda| < \epsilon\}| \ge \frac{1}{n}|\{k(j) \le n\}| - \frac{1}{n}\, O(1) \ge \frac{d}{2}$$

for infinitely many n. Hence, $\delta\{k \in \mathbb{N} : |x_k - \lambda| < \epsilon\} \ne 0$, which means that $\lambda \in \Gamma_x$.

Although our experience with ordinary limit points may lead us to expect that Λ_x and Γ_x are equivalent, the next example shows that this is not always the case. This completes the proof.

Example 3.2.3. *Define the sequence x by*

$$x_k = \ell/p, \quad \text{where } k = 2^{p-1}(2q+1),$$

i.e., $p - 1$ is the number of factors of 2 in the prime factorization of k. It is easy to see that for each p, $\delta\{k : x_k = \ell/p\} = 2^{-p} > 0$, whence $1/p \in \Lambda_x$. Also, $\delta\{k : 0 < x_k < \ell/p\} = 2^{-p}$, so $0 \in \Gamma_x$, and we have $\Gamma_x = \{0\} \cup \{\ell/p\}_{p=1}^{\infty}$. Now we assert that $0 \notin \Lambda_x$; for, if $\{x\}_K$ is a subsequence that has limit zero, then we can show that $\delta(K) = 0$. This is done by observing that for each p,

$$|K_n| = |\{k \in K_n : x_k \ge l/p\}| + |\{k \in K_n : x_k < l/p\}|$$

$$\le O(1) + |\{k \in \mathbb{N} : x_k < l/p\}| \le O(1) + n/2^p.$$

Thus $\delta(K) \le 2^{-p}$, and since p is arbitrary this implies that $\delta(K) = 0$.

It is easy to prove that if x is a statistically convergent sequence, say $st\text{-}\lim x = \lambda$, then Λ_x and Γ_x are both equal to the singleton set $\{\lambda\}$. The converse is not true, as one can see by taking $x_k = [1 + (-l)^k]k$. The following example presents a sequence x for which Γ_x is an interval while $\Lambda_x = \phi$.

Example 3.2.4. Let x be the sequence $\{0, 0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, \ldots.\}$. This sequence is uniformly distributed in [0, 1] (see [22]), so we have not only that $L_x = [0, 1]$ but also the density of the x_k's in any subinterval of length d is d itself. Therefore for any γ in [0, 1],

$$\delta\{k \in \mathbb{N} : x_k \in (\gamma - \epsilon, \gamma + \epsilon)\} \ge \epsilon > 0.$$

Hence, $\Gamma_x = [0, 1]$. On the other hand, if $\lambda \in [0, 1]$ and $\{x\}_K$ is a subsequence that converges to λ, then we claim that $\delta\{K\} = 0$. To prove this assertion, let $\epsilon > 0$ be given and note that for each n,

$$|K_n| \le |\{k \in K_n : |x_k - \lambda| < \epsilon\}| + |\{k \in K_n : |x_k - \lambda| \ge \epsilon\}|$$

$$\le 2\epsilon n + O(1).$$

Consequently, $\delta\{k(j)\} \leq 2\epsilon$, and since ϵ is arbitrary, we conclude that $\delta\{k(j)\} = 0$. Hence, $\Lambda_x = \phi$.

From Example 3.2.3 we see that Λ_x need not be a closed point set. The next result states that Γ_x, like L_x, is always a closed set.

Proposition 3.2.2. For any number sequence x, the set Γ_x of statistical cluster points of x is a closed point set.

Proof. Let p be an accumulation point of Γ_x, if $\epsilon > 0$ then Γ_x contains some point γ in $(p-\epsilon, p+\epsilon)$. Choose ϵ' so that $(\gamma - \epsilon', \gamma + \epsilon') \subseteq (p-\epsilon, p+\epsilon)$. Since $\gamma \in \Gamma_x$, $\delta\{k : x_k \in (\gamma - \epsilon', \gamma + \epsilon')\} \neq 0$, which implies that $\delta\{k : x_k \in (p-\epsilon, p+\epsilon)\} \neq 0$. Hence, $p \in \Gamma_x$.
This completes the proof.

For a given sequence x its statistical convergence or nonconvergence is not altered by changing the values of a thin subsequence ([16, Theorem 1]). We now show that the same is true for statistical limit points and cluster points.

Theorem 3.2.1. If x and y are sequences such that $x_k = y_k$ for almost all k, then $\Lambda_x = \Lambda_y$ and $\Gamma_x = \Gamma_y$.

Proof. Assume $\delta\{k : x_k \neq y_k\} = 0$ and let $\gamma \in \Lambda_x$, say $\{x\}_K$ is a nonthin subsequence of x that converges to λ. Since $\delta\{k : k \in K$ and $x_k \neq y_k\} = 0$, it follows that $\{k : k \in K$ and $x_k = y_k\}$ does not have density zero. Therefore the latter set yields a nonthin subsequence $\{y\}_{K'}$ of $\{y\}_K$ that converges to λ. Hence, $\lambda \in \Lambda_y$ and $\Lambda_x \subseteq \Lambda_y$. By symmetry we see that $\Lambda_y \subseteq \Lambda_x$, whence $\Lambda_x = \Lambda_y$. The assertion that $\Gamma_x = \Gamma_y$ is proved by a similar argument.
This completes the proof.

In the next theorem we establish a strong connection between statistical cluster points and ordinary limit points

Theorem 3.2.2. If x is a number sequence then there exists a sequence y such that $L_y = \Gamma_x$ and $y_k = x_k$ for almost all k; moreover, the range of y is a subset of the range of x.

Proof. If Γ_x is a proper subset of L_x, then for each ξ in $L_x \sim \Gamma_x$ choose an open interval I_ξ with center ξ such that $\delta\{k : x_k \in I_\xi\} = 0$. The collection of all such I_ξ's is an open cover of $L_x \sim \Gamma_x$, and by the Lindelöf Covering Property there exists a countable subcover, say $\{I_j\}_{j=1}^{\infty}$. Thus each I_j contains a thin subsequence of x. By a result of Connor [7, Corollary 9], this countable collection of sets, each having density zero, yields a single set Ω such that $\delta(\Omega) = 0$ and for each j, $\{k : x_k \in I_j\} \sim \Omega$ is a finite set. Let $\mathbb{N} \sim \Omega = \{j(k) : k \in \mathbb{N}\}$, and define the sequence y by

$$y_k = \begin{cases} x_{j(k)}, & \text{if } k \in \Omega \\ x_k, & \text{if } k \in \mathbb{N} \sim \Omega. \end{cases}$$

Obviously $\delta\{k : y_k \neq x_k\} = 0$ and Theorem 3.2.1 ensures that $\Gamma_y = \Gamma_x$. Since the subsequence $\{y\}_\Omega$ has no limit point that is not also a statistical limit point of y, it follows that $L_y = \Gamma_y$; hence, $L_y = \Gamma_x$.
This completes the proof.

Remark 3.2.1. The conclusion of Theorem 3.2.2 is not valid if Γ_x is replaced by Λ_x, because L_y is always a closed set while Λ_x need not be closed (as shown in Example 3.2.3).

3.3. Completeness theorems for statistical convergence

There are several well known theorems that are equivalent to the completeness of the real number system. When such a theorem concerns sequences we can attempt to formulate and prove a statistical analogue of that theorem by replacing ordinary limits with statistical limits. For example, in [16, Theorem 1] it is proved that a number sequence is statistically convergent if and only if it is a statistically Cauchy sequence. A sequential version of the Least Upper Bound Axiom (in $\mathbb{R}$) is the Monotone Sequence Theorem: if the (real) number sequence x is nondecreasing and bounded above, then x is convergent. The following result, which is an easy consequence of [16, Theorem 1], is a statistical analogue of that theorem.

Proposition 3.3.1. Suppose x is a number sequence and $M = \{k \in \mathbb{N} : x_k \leq x_{k+1}\}$; if $\delta\{M\} = 1$ and x is bounded on M, then x is statistically convergent.

Another completeness result for $\mathbb{R}$ is the Bolzano-Weierstrass Theorem which asserts that $L_x \neq \phi$ for a bounded sequence x. Example 3.2.4 shows that a bounded sequence might have $\Lambda_x = \phi$, but there is an analogue of the Bolzano-Weierstrass Theorem that uses statistical cluster points.

Theorem 3.3.1. If x is a number sequence that has a bounded nonthin subsequence, then x has a statistical cluster point.

Proof. Given such an x, Theorem 3.2.2 ensures that there exists a sequence y such that $L_y = \Gamma_x$ and $\delta\{k \in \mathbb{N} : y_k \neq x_k\} = 0$. Then y must have a bounded nonthin subsequence, so by the Bolzano-Weierstrass Theorem $L_y \neq \phi$, whence $\Gamma_x \neq \phi$.

Corollary 3.3.1. If x is a bounded number sequence, then x has a statistical cluster point.

The next result is a statistical analogue of the Heine-Borel Covering Theorem. If x is a bounded number sequence, let $\bar{X}$ denote the compact set $\{x_k : k \in \mathbb{N}\} \cup L_x$. A sequential version of the Heine-Borel Theorem tells us that if $\{J_n\}$ is a collection of open sets that covers $\bar{X}$, then there is a finite subcollection of $\{J_n\}$ that covers $\bar{X}$. To form a statistical analogue of this result we replace L_x with Γ_x and define the set

$$\mathrm{X} = \{x_k : k \in \mathbb{N}\} \cup \Gamma_x,$$

which we might call the statistical closure of x. It is easy to see that X need not be a closed set; indeed, X is a closed set if and only if X equals $\{x_k : k \in \mathbb{N}\} \cup L_x$, the ordinary closure of x.

Theorem 3.3.2. If x is a bounded number sequence, then it has a thin subsequence $\{x\}_K$ such that $\{x_k : k \in \mathbb{N} \sim K\} \cup \Gamma_x$ is a compact set.

Proof. Using Theorem 3.2.2 we can choose a bounded sequence y such that $L_y = \Gamma_x$, $\{y_k : k \in \mathbb{N}\} \subseteq \{x_k : k \in \mathbb{N}\}$ and $\delta(K) = 0$, where $K = \{k \in \mathbb{N} : x_k \neq y_k\}$. This yields

$$\{x_k : k \in \mathbb{N} \sim K\} \cup \Gamma_x = \{y_k : k \in \mathbb{N}\} \cup L_y.$$

and the right-hand member is a compact set.

It is easy to see that the proof of Theorem 3.3.2 remains valid even for unbounded x provided that x is bounded for almost all k, i.e., there is a thin sequence $\{x\}_M$ such that $\{x_k : k \in \mathbb{N} \sim M\}$ is a bounded set.

Finally, we note that for the compact set in Theorem 3.3.2 we cannot use Λ_x in place of Γ_x. In Example 3.2.3, $\Lambda_x = \{1/p : p \in \mathbb{N}\}$ and for each p in $\mathbb{N}$, $\delta\{k \in \mathbb{N} : x_k = l/p\} = 2^{-p}$. If $\{x\}_K$ is any thin subsequence then for each p, $\delta\{k \in \mathbb{N} \sim K : x_k = l/p\} = 2^{-p}$ and therefore $\{x_k : k \in \mathbb{N} \sim K\}$ still has zero as a limit point. Consequently, $\{x_k : k \in \mathbb{N} \sim K\} \cup \Lambda_x$ is not compact.

3.4. Statistical limit superior and limit inferior

Throughout the chapter k and n will always denote positive integers; x, y, and z will denote real number sequences; and $\mathbb{N}$ and $\mathbb{R}$ will denote the sets of positive integers and real numbers respectively. If $K \subseteq \mathbb{N}$, then $K_n = \{k : k \leq n\}$ and $|K_n|$, denotes the cardinality of K_n.

For a real number sequence x let B_x denote the set:

$$B_x = \{b \in \mathbb{R} : \delta\{k : x_k > b\} \neq 0\};$$

similarly,

$$A_x = \{a \in \mathbb{R} : \delta\{k : x_k < a\} \neq 0\}.$$

Note that throughout this chapter the statement $\delta\{K\} \neq 0$ means that either $\delta\{K\} > 0$ or K does not have natural density.

Definition 3.4.1. If x is a real number sequence, then the *statistical limit superior of x is given by*

$$st\text{-}\limsup x = \begin{cases} \sup B_x, & \text{if } B_x \neq \phi \\ -\infty, & \text{if } B_x = \phi. \end{cases}$$

Also, the *statistical limit inferior* of x is given by

$$st\text{-}\liminf x = \begin{cases} \inf A_x, & \text{if } A_x \neq \phi \\ +\infty, & \text{if } A_x = \phi. \end{cases}$$

A simple example will help to illustrate the concepts just defined. Let the sequence x be given by

$$x_k = \begin{cases} k, & \text{if } k \text{ is an odd square} \\ 2, & \text{if } k \text{ is an even square} \\ 1, & \text{if } k \text{ is an odd nonsquare} \\ 0, & \text{if } k \text{ is an even nonsquare.} \end{cases}$$

Note that although x is unbounded above, it is "statistically bounded" because the set of squares has density zero. Thus $B_x = (-\infty, 1)$ and $st\text{-}\limsup x = 1$. Also, x is not statistically convergent since it has two (disjoint) subsequences of positive density that converge to 0 and 1, respectively. (See [16], Theorem 1). Also note that the set of statistical cluster points of x is $\{0, 1\}$ and $st\text{-}\limsup x$ equals the greatest element while st-lim inf is the least element of this set. This observation suggests the main idea of the first theorem, which can be proved by a straightforward least upper bound argument.

Theorem 3.4.1. If $\beta = st\text{-}\limsup x$ is finite, then for every positive number ϵ,

$$\delta\{k : x_k > \beta - \epsilon\} \neq 0 \quad \text{and} \quad \delta\{k : x_k > \beta + \epsilon\} = 0. \tag{3.4.1}$$

Conversely, if (3.4.1) holds for every positive ϵ, then $\beta = st\text{-}\limsup x$.

The dual statement for $st\text{-}\liminf x$ is as follows.

Theorem 3.4.2. If $\alpha = st\text{-}\liminf x$ is finite, then for every positive number ϵ,

$$\delta\{k : x_k < \alpha + \epsilon\} \neq 0 \quad \text{and} \quad \delta\{k : x_k < \alpha - \epsilon\} = 0. \tag{3.4.2}$$

Conversely, if (3.4.2) holds for every positive ϵ then $\alpha = st\text{-}\liminf x$.

From the definition of statistical cluster point in [17] we see that Theorems 3.2.1 and 3.2.2 can be interpreted as saying that $st\text{-}\limsup x$ and $st\text{-}\liminf x$ are the greatest and least statistical cluster points of x. The next theorem reinforces that observation.

Theorem 3.4.3. For any sequence x, $st\text{-}\liminf x \leq st\text{-}\limsup x$.

Proof. First consider the case in which $st\text{-}\limsup x = -\infty$. This implies that $B_x = \phi$, so for every b in $\mathbb{R}$, $\delta\{k : x_k > b\} = 0$. This implies that $\delta\{k : x_k \leq b\} = 1$, so for every a in $\mathbb{R}$, $\delta\{k : x_k < a\} \neq 0$. Hence, $st\text{-}\liminf x = -\infty$.

The case in which $st\text{-}\limsup x = +\infty$ needs no proof, so we next assume that $\beta = st\text{-}\limsup x$ is finite, and let $\alpha = st\text{-}\liminf x$. Given $\epsilon > 0$ we show that $\beta + \epsilon \in A_x$, so that $\alpha \leq \beta + \epsilon$. By Theorem 3.4.1, $\delta\{k : x_k > \beta + \frac{\epsilon}{2}\} = 0$ because $\beta = lubB_x$. This implies that $\delta\{k; x_k \leq \beta + \frac{\epsilon}{2}\} = 1$ which, in turn, implies that $\delta\{k : x_k < \beta + \epsilon\} = 1$. Hence, $\beta + \epsilon \in A_x$. By definition $\alpha = \inf A_x$, so we conclude that $\alpha \leq \beta + \epsilon$.; and since ϵ is arbitrary this gives us $\alpha \leq \beta$.

From Theorem 3.2.3 and the above definition, it is clear that

$$\liminf x \leq st\text{-}\liminf x \leq st\text{-}\limsup x \leq \limsup x \tag{3.4.3}$$

for any sequence x.

A statistical limit point of a sequence x is defined in [17] as the limit of a subsequence of x whose indices do not have zero density. Since it was noted there that a bounded sequence might have no statistical limit point, one cannot say that

st- $\limsup x$ is equal to the greatest such point ([17], Example 4). This suggests the following question:

If x does have a greatest statistical limit point μ, does it follow that $\mu = st\text{-}\limsup x$?

The answer is no, which is shown by the following sequence:

Example 3.4.1. Following ([17], Example 4), we let u be the uniformly distributed sequence $u = \{0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, ...\}$ and define

$$x_{2k-1} = 0 \text{ and } x_{2k} = u_k.$$

Then $st\text{-}\limsup x = 1$ because $\delta\{k : x_k > 1 - \epsilon\} = \epsilon/2$. Also, zero is the only statistical limit point of x because u has none (as shown in [17]). Hence, the greatest statistical limit point of x is zero, but $st\text{-}\limsup x = 1$.

The next result is another statistical analogue of a very basic property of convergent sequences. For clarity of presentation we first give a formal definition of another statistical concept.

Definition 3.4.2. The real number sequence x is said to be statistically bounded if there is a number B such that $\delta\{k : |x_k| > B\} = 0$.

Note that statistical boundedness implies that st-lim sup and st-lim inf are finite, so Properties (3.4.1) and (3.4.2) of Theorems 3.4.1 and 3.4.2 hold.

Theorem 3.4.4. The statistically bounded sequence x is statistically convergent if and only if

$$st\text{-}\liminf x = st\text{-}\limsup x.$$

Proof. Let $\alpha = st\text{-}\liminf x$ and $\beta = st\text{-}\limsup x$. First assume that $st\text{-}\lim x = L$ and $\epsilon > 0$, then $\delta\{k : |x_k - L| \geq \epsilon\} = 0$, so $\delta\{k : x_k > L + \epsilon\} = 0$, which implies that $\beta \leq L$. We also have $\delta\{k : x_k < L - \epsilon\} = 0$ which implies that $L \leq \alpha$. Therefore $\beta \leq \alpha$, which we combine with Theorem 3.2.2 to conclude that $\alpha = \beta$.

Next assume $\alpha = \beta$ and define $L = \alpha$. If $\epsilon > 0$ then (3.4.1) and (3.4.2) of Theorems 3.4.1 and 3.4.2 imply $\delta\{k : x_k > L + \frac{\epsilon}{2}\} = 0$ and $\delta\{k : x_k < L - \frac{\epsilon}{2}\} = 0$. Hence, $st\text{-}\lim x = L$.

CHAPTER 4

STATISTICAL CORE

4.1. Introduction

The core of a complex number sequence as introduced by Knopp [21] is inherently connected to the set of limit points of the sequence. A variation of the concept of limit point based on statistical convergence was introduced in [16] and this led to a definition of the statistical core of a real number sequence in [15]. The present study uses a broader definition, similar to Knopp (see [18, p. 55]), which extends the previous definition of statistical core to include complex sequences. The main problem that is addressed in this chapter is to determine which matrix transformations map every bounded sequence into one whose core is a subset of the core of the original sequence. For the Knopp core this problem has been studied by many authors including [1, 2, 21, 26, 32]. The main result of this chapter gives necessary and sufficient conditions on a matrix A so that the Knopp core of Ax is contained in the statistical core of x for every bounded x. Since the statistical core is always a subset of the Knopp core, this yields sufficient conditions for the statistical core of Ax to be contained in the statistical core of x. The final section follows recent work of Choudhary [5] in giving conditions on matrices A and B so that the Knopp core of Ax is contained in the statistical core of Bx for every bounded x. This implies sufficient conditions for the statistical core of Ax to be a subset of the statistical core of Bx for every bounded x.

4.2. Statistical core of a real bounded sequence

In [21] Knopp introduced the concept of the core of a sequence and proved the well-known Core Theorem as mentioned in Chapter 1. Since the core of a bounded sequence x is the closed convex hull of the set of limit points of x, we can replace limit points with statistical cluster points to produce a natural analogue of Knopp's core.

Definition 4.2.1. If x is a statistically bounded sequence, then the *statistical core* of x is the closed interval $[st\text{-}\liminf x, st\text{-}\limsup x]$. In case x is not statistically bounded, *st-core*$\{x\}$ is defined accordingly as either $[st\text{-}\liminf x, \infty)$, $(-\infty, \infty)$, or $(-\infty, st\text{-}\limsup x]$.

We shall denote the statistical core of x by $st\text{-}core\{x\}$, and $K\text{-}core\{x\}$ will denote the usual core. It is clear from (3.4.3) that for any real sequence x

$$st\text{-}core\{x\} \subseteq K\text{-}core\{x\}.$$

Recall that the Core Theorem asserts that $K\text{-}core\{Ax\} \subseteq K\text{-}core\{x\}$, whenever Ax exists for the nonnegative regular matrix A[18, *p.* 55]. In [26] Maddox proves a variant of the Core Theorem that $\limsup Ax \leq \limsup x$ for every bounded x if and only if A is regular and $\lim_n \sum_{k=0}^{\infty} | a_{nk} |= 1$. We shall prove a similar result for the $st\text{-}core\{x\}$. For this purpose let us recall some previous results and notations. In [9] Connor proved that the set of bounded statistically convergent sequences is equal to the set of bounded strongly p-Cesáro summable sequences ($S \cap \ell_\infty = w_p \cap \ell_\infty$). In [28] Maddox proved that a matrix A maps $w_p \cap \ell_\infty$, into c if and only if A is in the class T^*, i.e., A is regular and $\lim_n \sum_{k \in E} |a_{nk}| = 0$ for every $E \subseteq \mathbb{N}$ such that $\delta\{E\} = 0$. Through out the following we shall use the abbreviations

$$\alpha(x) = st\text{-}\liminf x \quad \text{and} \quad \beta(x) = st\text{-}\limsup x.$$

Lemma 4.2.1. Suppose the matrix A satisfies $\lim_n \sum_{k=1}^{\infty} | a_{nk} |< \infty$; then

$$\limsup Ax \leq st\text{-}\limsup x, \quad \text{for every } x \in \ell_\infty \tag{4.2.1}$$

if and only if

$$A \in T^* \quad \text{and} \quad \lim_n \sum_{k=1}^{\infty} | a_{nk} |= 1. \tag{4.2.2}$$

Proof. Assume A satisfies (4.2.1) and $x \in \ell_\infty$, then $\beta(x) \leq \limsup x$ and since $\sup_n \sum_k | a_{nk} |< \infty$, $Ax \in \ell_\infty$. By (4) we have

$$-\beta(-x) \leq -\limsup(-Ax) \leq \limsup Ax \leq \beta(x),$$

or

$$st\text{-}\liminf x \leq \liminf Ax \leq \limsup Ax \leq \beta(x). \tag{4.2.3}$$

If $x \in S \cap \ell_\infty$ we have $\alpha(x) = \beta(x) = st\text{-}\lim x$, so (4.2.3) implies that $\lim Ax = st\text{-}\lim x$. Hence, A maps $S \cap \ell_\infty$ into c, so by the theorems of Maddox and Connor, $A \in T^*$. Also, since $\beta(x) \leq \limsup x$, (4.2.1) implies that $\limsup Ax \leq \limsup x$ and Maddox's variant of Knopp's Core Theorem yields

$$\lim_n \sum_{k=1}^{\infty} | a_{nk} |= 1.$$

Conversely, assume (4.2.2) and let x be bounded, then $Ax \in \ell_\infty$ and $\beta(x)$ is finite. Given $\epsilon > 0$, let $E = \{k : x_k > \beta(x) + \epsilon\}$. Thus $\delta\{E\} = 0$, and if $k \notin E$ then $x_k \leq \beta(x) + \epsilon$. For any real number z we write

$$z^+ = \max\{z, 0\} \text{ and } z^- = \max\{-z, 0\},$$

whence

$$|z| = z^+ + z^-, \quad z = z^+ - z^- \text{ and } |z| - z = 2z^-.$$

For a fixed positive integer m we write

$$\begin{aligned}
(Ax)_n &= \sum_{k<m} a_{nk}x_k + \sum_{k\geq m} a_{nk}x_k \\
&= \sum_{k<m} a_{nk}x_k + \sum_{k\geq m} a_{nk}^{+}x_k - \sum_{k\geq m} a_{nk}^{-}x_k \\
&\leq \|x\|_\infty \sum_{k<m} |a_{nk}| \;+\; \sum_{\substack{k\geq m \\ k\notin E}} a_{nk}^{+}x_k \;+\; \sum_{\substack{k\geq m \\ k\in E}} a_{nk}^{+}x_k \;+\; \|x\|_\infty \sum_{k\geq m} (|a_{nk}| - a_{nk}) \\
&= \|x\|_\infty \sum_{k<m} |a_{nk}| \;+\; (\beta(x) + \epsilon) \sum_{\substack{k\geq m \\ k\notin E}} |a_{nk}| \;+\; \|x\|_\infty \sum_{\substack{k\geq m \\ k\in E}} |a_{nk}| \\
&\qquad + \|x\|_\infty \sum_{k\geq m} (|a_{nk}| - a_{nk}).
\end{aligned}$$

Taking the limit superior as $n \to \infty$ and using (4.2.2) and the regularity of A, we get

$$\limsup (Ax)_n \;\leq\; \beta(x) + \epsilon.$$

Since ϵ is arbitrary, we conclude that (4.2.1) holds and the proof is complete.

It is clear that one can prove a similar result for $\alpha(x) \leq \liminf Ax$ and therefore we have the following result:

Theorem 4.2.1. (Statistical Core Theorem) If the matrix A satisfies $\sup_n \sum_{k=1}^{\infty} | a_{nk} | < \infty$, then

$$K\text{-}core\{Ax\} \subseteq st\text{-}core\{x\}, \text{ for every } x \text{ in } \ell_\infty$$

if and only if

$$A \in T^* \text{ and } \lim_n \sum_{k=1}^{\infty} | a_{nk} | = 1.$$

In [4] Buck proved that a sequence that is C_1-summable to its limit superior is statistically convergent. The next theorem is a statistical analogue of that result.

Theorem 4.2.2. If the sequence x is bounded above and C_1-summable to the number $\beta = st\text{-}\limsup x$, then x is statistically convergent to β.

Proof. Suppose that x is not statistically convergent to β. Then by Theorem 3.4.2, $st\text{-}\liminf x < \beta$, so there is a number $\mu < \beta$ such that $\delta\{k : x_k < \mu\} \neq 0$. Let $K' = \{k : x_k < \mu\}$. By the definition of β, $\delta\{k : x_k > \beta + \epsilon\} = 0$ for every $\epsilon > 0$. Define

$$K'' = \{k : \mu \leq x_k \leq \beta + \epsilon\} \text{ and } K''' = \{k : x_k > \beta + \epsilon\},$$

and let $B = \sup_k x_k < \infty$. Since $\delta\{K'\} \neq 0$, there are infinitely many n such that

$$\frac{1}{n}|K'_n| \geq d > 0, \tag{4.2.4}$$

and for each such n we have

$$\begin{aligned}
(C_1x)_n &= \frac{1}{n}\sum_{k\in K'_n} x_k + \frac{1}{n}\sum_{k\in K''_n} x_k + \frac{1}{n}\sum_{k\in K'''_n} x_k \\
&< \frac{\mu}{n}|K'_n| + \frac{\beta+\epsilon}{n}|K''_n| + \frac{B}{n}|K'''_n| \\
&= \mu\frac{|K'_n|}{n} + (\beta+\epsilon)\Big(1 - \frac{|K'_n|}{n}\Big) + \circ(1) \\
&\leq \beta - d(\beta - \mu) + \epsilon(1-d) + \circ(1).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary it follows that

$$\liminf C_1 x \leq \beta - d(\beta - \mu) < \beta.$$

Hence, x is not C_1-summable to β, which completes the proof.

By symmetry we have the dual result for lower bounds.

Corollary 4.2.1. If the sequence x is bounded below and C_1-summable to the number $\alpha = st\text{-}\liminf x$, then x is statistically convergent to α.

Since Buck's Theorem, which was the motivation for Theorem 4.2.2, does not assume an upper bound of the sequence, it is natural to ask if that hypothesis could be eliminated from Theorem 4.2.2. The following example shows that the upper bound cannot be omitted or even replaced by the weaker assumption of a statistical upper bound.

Example 4.2.1. Let x be the sequence given by

$$x_k = \begin{cases} \sqrt{k}, & \text{if } k \text{ is a square} \\ 0, & \text{if } k \text{ is an odd nonsquare} \\ 1, & \text{if } k \text{ is an even nonsquare.} \end{cases}$$

Since $\delta\{k : x_k = 0\} = 1/2 = \delta\{k : x_k = 1\}$, it is clear that $st\text{-}\liminf x = 0$ and $st\text{-}\limsup x = 1$. Therefore x is not statistically convergent. Also note that x is statistically bounded since $\delta\{k : |x_k| > 1\} = 0$. It remains to show that $C_1 x$ has $\lim 1 = st\text{-}\limsup x$. Let K^2 denote the set of squares and let K^0 and K^1 denote, respectively, the sets of odd and even nonsquares. With $[t] = \max\{k : k \le t\}$, this yields

$$\begin{aligned} (C_1 x)_n &= \frac{1}{n} \sum_{k \in K_n^0} x_k + \frac{1}{n} \sum_{k \in K_n^1} x_k + \frac{1}{n} \sum_{k \in K_n^2} x_k \\ &= 0 + \frac{1}{n} \frac{[n - [\sqrt{n}]]}{2} + \frac{1}{n} \sum_{i \le \sqrt{n}} i \\ &= 1 + o(1). \end{aligned}$$

4.3. Statistical core of complex sequences

In chapter 3 a statistical cluster point of a sequence x is defined as a number γ such that for every $\epsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - \gamma| < \epsilon\}$ does not have density zero. In [15] the sequence x is defined to be statistically bounded if x has a bounded subsequence of density one and the statistical core of such an x (of real values) is the closed interval $[st\text{-}\liminf x,\ st\text{-}\limsup x]$, where $st\text{-}\liminf x$ and $st\text{-}\limsup x$ are the least and greatest statistical cluster points of x. It is also known [15, Theorem 1] that, for a sequence x of real numbers, the number β is the $st\text{-}\limsup x$ if and only if for every $\epsilon > 0$,

$$\delta\{k : x_k > \beta - \epsilon\} \neq 0 \qquad \text{and} \qquad \delta\{k : x_k > \beta + \epsilon\} = 0.$$

We begin by giving a definition of the statistical core for complex sequences. It is convenient to use the terminology and notation of [16] in situations where a subsequence of density one satisfies a certain property. For example, if x and y are sequences such that

$$\delta\{k \in \mathbb{N} : x_k = y_k\} = 1,$$

then we write $x_k = y_k$ *a. a. k*, which can be read $x_k = y_k$ for almost all k.

Definition 4.3.1. For any complex sequence x let $\mathbb{H}(x)$ be the collection of all closed half-planes that contain x_k $a.\ a.\ k$, then the statistical core of x is given by

$$st\text{-}core\{x\} = \bigcap_{H\in\mathbb{H}(x)} H.$$

It should be noted that in the definition of the K-$core\{x\}$, the closed convex hull $C_n(x)$ is the intersection of all closed half planes that contain $\{x_k\}_{k\geq n}$; in defining the st-$core\{x\}$ we have simply replaced $\{x_k\}_{k\geq n}$ by an arbitrary subsequence of density one. Therefore, it follows that for all x, st-$core\{x\} \subseteq K$-$core\{x\}$. Also, it is easy to note that if x is a statistically bounded real sequence, then

$$st\text{-}core\{x\} = [st\text{-}\liminf x,\ st\text{-}\limsup x],$$

as in [15].

In order to prove the main theorem it is convenient to prove first an equivalent form of the st-$core\{x\}$ that is motivated by Shcherbakov's form [37] of the K-$core\{x\}$ for bounded x.

Lemma 4.3.1. Let x be a statistically bounded sequence, for each $z \in \mathbb{C}$ let

$$B_x(z) = \big\{w \in \mathbb{C} : |w - z| \leq st\text{-}\limsup |x_k - z|\big\},$$

then

$$st\text{-}core\{x\} = \bigcap_{z\in\mathbb{C}} B_x(z).$$

Proof. From the definition of st-$\limsup x$ and ([15],Theorem 1) we note that the disk $B_x(z)$ is equal to the intersection of all closed disks centered at z that contain x_k $a.\ a.\ k$. First assume $w \notin \bigcap_{z\in\mathbb{C}} B_x(z)$, say $w \notin B_x(z^*)$ for some z^*. Let H be the half-plane containing $B_x(z^*)$ whose boundary line is perpendicular to the line containing w and z^* and tangent to the circular boundary of $B_x(z^*)$. Since $B_x(z^*) \subset H$ and $B_x(z^*)$ contains x_k $a.\ a.\ k$, it follows that $H \in \mathbb{H}(x)$. Since $w \in H$, this implies $w \notin \bigcap_{H\in\mathbb{H}(x)} H$. Hence, st-$core\{x\} \subseteq \bigcap_{z\in\mathbb{C}} B_x(z)$.
Conversely, if $w \notin \bigcap_{H\in\mathbb{H}(x)} H$, let H be a plane in $\mathbb{H}(x)$ such that $w \notin H$. Let L be the line through w that is perpendicular to the boundary of H and let p be the midpoint of the segment of L between w and H. Let z be a point of L such that $z \in H$ and consider the disk

$$B(z) = \{\zeta \in \mathbb{C} : |\ \zeta - z\ |\ \leq\ |\ p - z\ |\ \}.$$

Since x is statistically bounded and $x_k \in H$ $a.\ a.\ k$, we can choose z sufficiently far from p so that $|\ p - z\ | = st$-$\limsup\ |\ x_k - z\ |$. Thus $B(z)$ is one of the $B_x(z)$ disks and since $w \notin B(z)$ we conclude that $w \notin \bigcap_{z\in\mathbb{C}} B_x(z)$. This establishes the converse set inclusion and completes the proof of the Lemma.

Remark 4.3.1. The alternative form of *st-core*$\{x\}$ given in the preceding Lemma is not necessarily valid if x is not statistically bounded. For example, if $x_k = k$ for all k, then x has no statistical cluster point and *st-core*$\{x\} = \phi$. But for any $z \in \mathbb{C}$, no disk of finite radius can contain x_k $a.\ a.\ k$, so $st\text{-}\limsup |x_k - z| = \infty$ and $B_x(z)$ is the whole plane $\mathbb{C}$, whence $\bigcap_{z\in} B_x(z) = \mathbb{C}$.

We are now prepared to describe those matrices that transform each bounded sequence x into a sequence whose core is a subset of the statistical core of x. Throughout the remainder of this chapter the set of bounded complex sequences will be denoted by ℓ^∞.

Theorem 4.3.1. If A satisfies $\sup_n \sum_k |a_{nk}| < \infty$, then *K-core*$\{Ax\} \subseteq$ *st-core*$\{x\}$ for every $x \in \ell^\infty$ if and only if the following conditions hold:

(i) $A \in \mathbb{F}^*$, i.e., A is regular and $\lim_n \sum_{k\in E} |a_{nk}| = 0$ whenever $\delta(E) = 0$;

(ii) $\lim_n \sum\limits_{k=1}^{\infty} |a_{nk}| = 1$.

Proof. (I)(Necessity) If x is statistically convergent to L, then

$$\{L\} = st\text{-}core\{x\} \supseteq K\text{-}core\{Ax\}.$$

Since Ax is bounded for all $x \in \ell^\infty$, we must have that

$$K\text{-}core\{Ax\} = \{L\},$$

so Ax is convergent to L. Now the theorems of Connor [9] and Maddox [28] imply that A is regular and $\lim_n \sum_{k\in E} |a_{nk}| = 0$ whenever $\delta(E) = 0$, i.e., condition (i) holds. Also, we have

$$K\text{-}core\{Ax\} \subseteq st\text{-}core \subseteq K\text{-}core\{x\}.$$

Now ([32],Theorem 2.1) for the case $\alpha = 1$ and $\mathbb{K} = \mathbb{C}$ implies that $\limsup_n \sum\limits_{k=1}^{\infty} |a_{nk}| \leq 1$ and this is equivalent to $\lim_n \sum\limits_{k=1}^{\infty} |a_{nk}| = 1$, because A is regular. This proves the necessity of (i) and (ii).

(II) (Sufficiency) Assume (i) and (ii) and let $w \in$ *K-core*$\{Ax\}$. For any $z \in \mathbb{C}$ we have

$$
\begin{aligned}
| w - z | &\leq \limsup_n \left|z - (Ax)_n\right| \\
&= \limsup_n \left|z - \sum_{k=1}^{\infty} a_{nk} x_k\right| \\
&\leq \limsup_n \left|\sum_{k=1}^{\infty} a_{nk}(z - x_k)\right| + \limsup_n | z | \left|1 - \sum_{k=1}^{\infty} a_{nk}\right| \\
&= \limsup_n \left|\sum_{k=1}^{\infty} a_{nk}(z - x_k)\right|. \qquad (4.3.1)
\end{aligned}
$$

Let $r = st\text{-}\limsup |x_k - z|$, suppose $\epsilon > 0$ and let $E = \{k : |z - x_k| > r + \epsilon\}$, then $\delta(E) = 0$ and we write

$$\left|\sum_{k=1}^{\infty} a_{nk}(z - x_k)\right| \leq \sup_k |z - x_k| \sum_{k \in E} |a_{nk}| + (r + \epsilon) \sum_{k \notin E} |a_{nk}|.$$

Now (i) and (ii) imply that

$$\limsup_n \left|\sum_{k=1}^{\infty} a_{nk}(z - x_k)\right| \leq r + \epsilon.$$

From (4.3.1) we conclude that $|w - z| \leq r + \epsilon$ and since ϵ is arbitrary this yields $| w - z | \leq r$. Hence, $w \in B_x(z)$, so by the Lemma 4.3.1, $w \in$ *st-core*$\{x\}$. This completes the proof.

As noted above the *st-core* of any sequence is a subset of the K-*core*; therefore the preceding theorem gives an immediate corollary, whose conclusion is an exact statistical analogue of Knopp's original Kernsatz ([18], [21]).

Corollary 4.3.1. If the matrix A satisfies $\sup_n \sum_{k=1}^{\infty} | a_{nk} | < \infty$ and properties (i) and (ii) of Theorem 4.3.1, then

$$st\text{-}core\{Ax\} \subseteq st\text{-}core\{x\},$$

for every $x \in \ell^{\infty}$.

Unlike the theorem that precedes it, Corollary 4.3.1 does not have a valid converse. This is seen by the following example.

Example 4.3.1. We shall define A so that $(Ax)_n = x_n \ \ a.\ a.\ \ k$. By Theorem 1 of [17] this ensures that the statistical cluster points of Ax are the same as those of x, so $st\text{-}core\{Ax\} = st\text{-}core\{x\}$.

$$a_{nk} = \begin{cases} 1, & \text{if } k = n \text{ and } n \text{ is a nonsquare} \\ 1, & \text{if } k \ \leq \ n \text{ and } n \text{ is a square} \\ 0, & \text{otherwise,} \end{cases}$$

then

$$(Ax)_n = \begin{cases} x_n, & \text{if } n \text{ is a nonsquare} \\ \sum\limits_{k=1}^{n} x_k, & \text{if } n \text{ is a square.} \end{cases}$$

Observe that $\sup_n \sum\limits_{k=1}^{\infty} |a_{nk}| = \infty$, A neither satisfy (ii), nor does $\lim_n \sum_{k \in E} \mid a_{nk} \mid = 0$ whenever $\delta(E) = 0$.

Remark 4.3.2. It would be good to have necessary and sufficient conditions for a matrix A to yield $st\text{-}core\{Ax\} \subseteq st\text{-}core\{x\}$ for every $x \in \ell^{\infty}$, but for the present this remains an open question.

Since Theorem 4.3.1 and Corollary 4.3.1 are statistical analogues of previous theorems about the K-core, it is natural to ask if the restriction to bounded sequences can be replaced by statistical boundedness. This is not possible as the next example shows.

Example 4.3.2. Take $A = C_1$, the Cesáro (arithmetic) means and define x by

$$x_n = \begin{cases} \sqrt{k}, & \text{if } k \text{ is a square} \\ 0, & \text{otherwise.} \end{cases}$$

Then $x_k \ \ a.\ a.\ k$, so x is statistically bounded and $st\text{-}core\{x\} = \{0\}$. It is clear that C_1 satisfies all the conditions of A in Theorem 4.3.1, but

$$(C_1 x)_n = \frac{1}{n} \sum_{k \leq \sqrt{n}} k = \frac{1}{n} \Big[\frac{\sqrt{n}(\sqrt{n}+1)}{2} + 0(\sqrt{n}) \Big].$$

thus $\lim(C_1x)_n = 1/2$, so $st\text{-}core\{C_1x\} = K\text{-}core\{C_1x\} = \{1/2\}$.

In [5] Choudhary extended Knopp's Core Theorem to the case in which the cores of two transformations are compared, i.e., the conclusion is

$$K\text{-}core\{Ax\} \subseteq K\text{-}core\{Bx\},$$

so that replacing B by the identity matrix yields Knopps Theorem. In this section we prove a statistical analogue of Choudhary's Theorem.

Theorem 4.3.2. Let B be a normal matrix (i.e., triangular with nonzero diagonal entries), and denote its triangular inverse by $B^{-1} = [b_{nk}^{-1}]$. For an arbitrary matrix A, in order that, whenever $Bx \in \ell^\infty$, Ax should exist and be bounded and satisfy

$$K\text{-}core\{Ax\} \subseteq K\text{-}core\{Bx\} \tag{4.3.2}$$

it is necessary and sufficient that the following conditions hold:

(i) $C = AB^{-1}$ exists

(ii) $C \in \mathbb{F}^*$

(iii) $\lim_n \sum_{k=1}^{\infty} |c_{nk}| = 1$

(iv) for any fixed n,

$$\lim \sum_{k=0}^{v} \Big| \sum_{j=v+1}^{\infty} a_{nj} b_{jk}^{-1} \Big| = 0.$$

Proof. (I) (Necessity) If $(Ax)_n$ exists for every n whenever $Bx \in \ell^\infty$, then by ([5], Lemma 2) it follows immediately that (i) and (iv) hold. By that same Lemma we also have $Ax = Cy$, where $y = Bx$. Since $Ax \in \ell^\infty$ we have $Cy \in \ell^\infty$. Therefore (4.3.2) implies that $K\text{-}core\{Ax\} \subseteq K\text{-}core\{Bx\}$. Now Theorem 4.3.1 implies that (ii) and (iii) hold.

(II) (Sufficiency) Properties (i)-(iv) obviously imply the four conditions of Choudhary's Lemma 2 in [5], so it follows by the Lemma that $Cy \in \ell^\infty$, hence $Ax \in \ell^\infty$. Now Theorem 4.3.1 implies that K-core$\{Cy\} \subseteq$ K-core$\{y\}$ and since $y = Bx$ and $Cy = Ax$ we have K-core$\{Ax\} \subseteq$ K-core$\{Bx\}$.

As above, the fact that $st\text{-}core\{Ax\} \subseteq K\text{-}core\{Ax\}$ gives us an immediate corollary.

Corollary 4.3.2. If A and B satisfy conditions (i)-(iv) of Theorem 4.3.2, then

$$st\text{-}core\{Ax\} \subseteq st\text{-}core\{Bx\}$$

for every x such that $Bx \in \ell^\infty$.

The converse of Corollary 4.3.2 is false, as can be seen by Example 4.3.1 Similarly, since Theorem 4.3.2 reduces to Theorem 4.3.1 in case B is the identity, Example 4.3.2 shows that it is not possible to extend Theorem 4.3.2 from ℓ^∞ to statistically bounded sequences.

CHAPTER 5

STRONG p-CESÁRO CONVERGENT OF SEQUENCES

5.1. Introduction

The history of strong p-Cesáro summability, being longer, is not so clear. Strong 1-Cesàro summability appears to have been introdued as early as 1913 by Hardy and Littlewood [19] in relation to the convergence of a Fourier series. The concept does not appear to have been studied purely as a summability method until 1946 in an article by Kutter [23]. Maddox noted that w_p can be considered as a BK space if $1 \leq p < \infty$ and as a p-normable space if $0 < p < 1$ ([24], [25]) and at that time characterized the matrices which map w_p into c for $0 < p < \infty$. The concept has been generalized often and is often studied from the viewpoint of functional analysis.

5.2. Inclusion and equivalent theorems

In this section we articulate the promised connections between strong p-Cesàro summability and statistical convergence.

Theorem 5.2.1. Let $q \in \mathbb{R}$, $0 < q < \infty$. If a sequence is strongly q-Cesàro summable to L, then it is statistically convergent to L. If a bounded sequence is statistically convergent to L, then it is strongly q-Cesàro summable to L.

Proof. Observe that for any $x = \{x_n\} \in \omega$ and $\epsilon > 0$, we have that

$$\sum_{k=1}^{n} |x_k - L|^q \geq \left|\{k \leq n : |x_k - L|^q \geq \epsilon\}\right| \epsilon^q.$$

It follows that if x is srongly q-Cesáro summable to L then x is statistically convergent to L.

Now suppose that x is bounded and statistically convergent to L and set $k = \|x\|_\infty + \mid L \mid$. Let $\epsilon > 0$ be given and select N_ϵ such that

$$n^{-1}\left|k \leq n : |x_k - L| \geq (\epsilon/2)^{1/p}\right| < \epsilon/2K^p,$$

for all $n > N_\epsilon$ and set $L_n = \left\{k \leq n : |x_k - L| \geq (\epsilon/2)^{1/p}\right\}$. Now for $n > N_\epsilon$ we have that

$$
\begin{aligned}
(1/n)\sum_{k=1}^{n} \mid x_k - L \mid^p &= (1/n)\{ \sum_{k\in L_n} |x_k - L|^p + \sum_{\substack{k\notin L_n \\ k\le n}} |x_k - L|^p\} \\
&< (1/n)(n\epsilon/2k^p)k^p + (1/n)(n)(\epsilon/2) \\
&= \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}
$$

Hence x is strongly p-Cesáro summable to L.

The following corollary is an extention of a result of Maddox.

Corollary 5.2.1. Let $p, q \in \mathbb{R}$, $0 \le p < q < \infty$. Then $w_p \supseteq w_q$ and $w_p \cap \ell_\infty = w_q \cap \ell_\infty$.

For positive values of p and q both the inclusion $w_p \supseteq w_q$ ([23], [27]) (as a direct consequence of Hölder's inequality) and the equality $w_p \cap \ell_\infty = w_q \cap \ell_\infty$ [28] have already been established. Theorem 5.2.1 extends these results to the case $p = 0$ and $q > 0$.

The above result could also have been established as a consequence of corollary 4.11 of [12] (where, as a consequence of a deeper results, it can be shown that $w_1 \cap \ell_\infty = w_0 \cap \ell_\infty$) and Maddox's remark [28].

Similar but not identical versions of next results have occured frequently in the literature on statistical convergence and have been established independently by Salat [36], Fridy [16], and the Cannor [8]. Theorem 5.2.1 extends these results to strong p-Cesàro convergence.

Theorem 5.2.2.(Decomposition Theorem) If $x \in \omega$ is stongly p-Cesáro summable or statistically convergent to L, then there is a convergent sequence y and a statistically null sequence z such that y is convergent to L, $x = y + z$ and $\lim_n n^{-1}|\{k \le n : z_k \ne 0\}| = 0$. Moreover, if x is bounded and

$$\|z\|_\infty \le \|x\|_\infty + |L|.$$

Proof. First observe that if x is strongly p-Cesáro summable to L then x is statistically convergent to L. Now let $N_0 = 0$ and select an increasing sequence of positive integers $N_1 < N_2 < N_3 < \ldots$ such that if $n > N_j$ we have that

$$n^{-1}\big|\{k \le n : |x_k - L| \ge j^{-1}\}\big| < j^{-1}$$

We define y and z as follows: if $N_0 < k \le N_1$ set $z_k = 0$ and $y_k = x_k$. Now suppose that $j \ge 1$ and that $N_j < k \le N_{j+1}$. If $|x_k - L| < j^{-1}$ we set $y_k = x_k$ and $z_k = 0$ and if $|x_k - L| \ge j^{-1}$ we set $y_k = L$ and $z_k = x_k - L$. It is clear from our construction that $x = y + z$ and that $\|z\|_\infty \le \|x\|_\infty + |L|$ if x is bounded.

We claim that $\lim_k y_k = L$. Let $\epsilon > 0$ and pick j such that $\epsilon > j^{-1}$. Observe that for $k > N_j$ we have that $|y_k - L| < \epsilon$ since $|y_k - L| = |x_k - L| < \epsilon$ if $|x_k - L| < j^{-1}$ and $|y_k - L| = |L - L| = 0$ if $|x_k - L| > j^{-1}$. Since ϵ was arbitrary, we have established the the claim.

Next we claim that z is statistically null. First note that it suffices to show that $\lim_n n^{-1}|\{k \leq n : z_k \neq 0\}| = 0$ in order to establish the claim. This follows from observing that $|\{k \leq n : z_k \neq 0\}| \geq \big|\{k \leq n : |z_k| \geq \epsilon\}\big|$ for any natural number n and $\epsilon > 0$.

We now show that if $\delta > 0$ and $j \in N$ such that $j^{-1} < \delta$, then

$$|\{k \leq n : z_k \neq 0\}| < \delta, \quad \text{for all } n > N_j.$$

Recall from the construction that if $N_j < k \leq N_{j+1}$, then $z_k \neq 0$ only if $| x_k - L |> j^{-1}$. It follows that if $N_\ell < k \leq N_{\ell+1}$, then

$$\{k \leq n : z_k \neq 0\} \subseteq \big\{k \leq n : |x_k - L| > \ell^{-1}\big\}.$$

Consequently, if $N_\ell < n \leq N_{\ell+1}$ and $\ell > j$, then

$$n^{-1}|\{k \leq n : z_k \neq 0\}| \leq n^{-1}\big|\{k \leq n : |x_k - L| > \ell^{-1}\}\big| < \ell^{-1} < j^{-1} < \delta.$$

Which establishes the claim and hence the theorem.

The following corollary is immediately evident from the preceeding theorem.

Corollary 5.2.2. Let $x \in \omega$. If x is strongly p-Cesáro summable to L or statistically convergent to L, then x has a subsequence which converges to L.

The above corollary can be used to show there are bounded Cesàro summable sequences which are not statistically convergent. For instance the sequence $\{0, 1, 0, 1, 0, 1, ...\}$ is Cesàro summable to $1/2$ but fails to have any subsequence which converge to $1/2$ and hence can not be statistically convergent.

Buck introduced the concept of a sequence being convergent "for almost all n" as a special case of sequence being convergent "in density" in [4] as follows:
Let $A \subseteq \mathbb{N}$ and define $D(A)$ by

$$D(A) = \lim_n n^{-1}\big|\{k \leq n : k \in A\}\big|.$$

When the limit exists (note that Steinhaus theorem indicates that $D(A)$ is not defined for every $A \subseteq \mathbb{N}$). A sequence x is said to converge to L for almost all n (in the sense of Buck) if there is a set $A \subseteq \mathbb{N}$, $D(A) = 0$, such that for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for every $n \geq N$, $n \notin A$, $|x_n - L| < \epsilon$.

A straightforward argument shows that if a sequence is convergent to L for almost all n, then the sequence is statistically convergent to L. Theorem 5.2.2 asserts that statistically convergent sequences are convergent for almost all n.

We may now include the following results of Buck (Theorem 3.2, [4]) as a partial converse of corollary 5.2.2.

Proposition 5.2.1. Let $x \in \omega$. If $\liminf x_n = L$ and x is Cesáro summable to L, then x is statistically convergent to L.

We are also in position to record some Tauberian theorems of Fridy in a broader setting than they originally occured. For a sequence $x \in \omega$ we use the notation Δx to denote the sequence of forward differences :$\Delta x_k = x_k - x_{k+1}$. Fridy [16] established the following two results for the case of statistical convergence.

Corollary 5.2.3. Let $x \in \omega$. If x is statistically convergent to L or strongly p-Cesáro summable to L and $\Delta x_k = 0(1/k)$, then x convergent to L.

Corollary 5.2.4. Let $x \in \omega$, $\{k_i\}$ be an increasing sequence of positive integers such that $\liminf_i(k_{i+1}/k_i) > 1$ and suppose that x is a corresponding gap sequence: $\Delta x_k = 0$ if $k = k_i$ for $i \in \mathbb{N}$. If x is strongly p-Cesàro summable or statistically convergent to L, then x is convergent to L.

The proofs of the above two corollaries follow immediately from Fridy's results and theorem 5.2.1. It should also be noted that Fridy has shown that Corollary 5.2.3 gives the best possible Tauberian condition of order type for statistical convergence.

5.3. Matrix summability results

In this section we characterize the summability matrices which are stronger than bounded statistical convergence on the bounded sequences. We also show that the matrices which map all statistically convergent sequences into convergent sequences are trivial in that they must map every sequence into a convergent sequence. The former result is applied to Nörlund and Nöriund-type summability methods at end of this section.

We actually show more than that if $w_0 \subseteq c_A$ then $c_A = \omega$. We show that the only locally convex *FK space* that contains the statistically convergent sequences is ω. It follows from this results since there are sequences which are not statistically convergent, that w_0 cannot be given a locally convex *FK topology.* The chief tool in establishing the above claim is the following theorem of Bennett and Kalton:

Theorem 5.3.1. Let S be dense subspace of ω. The following are equivalent:

(a) S is barrelled

(b) If E is a locally convex FK *space* that contains S, then $E = \omega$.

Note should be taken that the above is a restricted version of the result that appears in [3]. There is also an exposition of this result in [40].

Lemma 5.3.1. The statistically convergent sequences form a dense, barrelled subspace of ω.

Proof. First recall that w_0 is barrelled if and only if every $\sigma(\phi,\ w_0)$ bounded subset of ϕ is $\sigma(\phi,\ \omega)$ bounded ([40], *p.* 310). We show that if $B \subseteq \phi$ is not $\sigma(\phi,\ \omega)$ bounded then B is not $\sigma(\phi,\ w_0)$ bounded and hence w_0 is a barrelled subspace of ω.

First note that we may assume that is a sequence of integers $\{k_n\}$ such that if $x \in B$ and $\text{supp}(x) = \{k \in \mathbb{N} : x_k \neq 0\} \subseteq \{1, 2, 3, ..., n\}$, then $\sup_{1 \leq i \leq n} \mid x_j \mid \leq k_n$. (Observe that if B falls this property then B is $\sigma(\phi, \phi)$ bounded and hence not $\sigma(\phi, w_0)$ bounded). Now let $y \in \omega$ such that $\sup_{x \in B} \mid \sum\limits_{i=1}^{\infty} x_i y_i \mid = \infty$ and set $y^{(n)} = \sum\limits_{i=1}^{n} y_i e^i$. Now, by the assumption we have that

$$\mid \sum_{i=1}^{n} x_i y_i \mid \leq n k_n \|y^{(n)}\|_\infty$$

for all $n \in \mathbb{N}$ and $x \in B$ such that $\text{supp}(x) \subseteq \{1, 2, 3, ..., n\}$.

Now select $x^1 \in B$ such that $\mid \sum\limits_{i=1}^{\infty} x_i^1 y_i \mid > k_1 \|y^1\|_\infty$ and select $j_1 > 1$ such that $x_{j_1} \neq 0$. Now assume that $\{x^1, x^2, x^3, ..., x^n\}$ have been selected and $j_1 < j_2 < j_3 < ... < j_n$ have been selected such that $(x^n)_{j_n} \neq 0$ and $j_n > \max\{n^2,\ \text{supp}(x^{n-1})\}$. Select x^n as follows: set $\ell = \max\{\text{supp}(x^n), (n+1)^2\}$ and pick x^{n+1} such that $\mid \sum\limits_{i=1}^{\infty} x_i^{n+1} y_i \mid > \ell k_\ell \|y^{(\ell)}\|_\infty$. Pick j_{n+1} such that $\ell < j_{n+1}$ and $(x^{n+1})_{j_{n+1}} \neq 0$. Proceed inductively. It is easy to select a sequence $\{\alpha_i\}$ such that $\sum\limits_{k=1}^{n} \alpha_k x_{j_k}^n$ tends to ∞ as n tends to ∞. Define a sequence z by $z_{j_k} = \alpha_k$ and $z_k = 0$ if $k \neq j_i$ for all $i \in \mathbb{N}$. Note that, since $j_n > n^2$ for all n, $z \in w_0$ and $\sum\limits_{k=1}^{\infty} (x^n)_k z_k = \sum\limits_{k=1}^{n} (x^n)_{j_k} \alpha_k$. It follows that B is not $\sigma(\phi,\ w_0)$ bounded and hence that w_0 is a barrelled subspace of ω.

Since $\phi \subseteq w_0$, it follows that w_0 is a dense barrelled subspace of ω.

The following result is now an immediate consequence of theorem 5.3.1

Theorem 5.3.2. If E is a locally convex FK space such that $w_0 \subseteq E$, then $E = \omega$.

The above theorem extends a result of Fridy [16] and includes a result of the Cannor [8] as is noted in the following corollary.

Corollary 5.3.1. $w_0 \subseteq c_A$ if and only if A has finitely many nonzero convergent columns.

Proof. If A has finitely many nonzero columns, then $c_A = \omega$ and consequently $w_0 \subseteq c_A$.
Conversely, if $w_0 \subseteq c_A$, then Theorem 5.3.2 asserts that $c_A = \omega$ since c_A is a locally convex FK *space*. The conclusion now follows trivially.

Maddox has characterized the matrices which have the property that $w_p \subseteq c_A$ for $0 < p < \infty$ in [24] and [25]. In particular he has shown as an extension of Kuttner's theorem [23], that if a matrix A has the property that $w_p \subseteq c_A$ and $0 < p < 1$, then $\ell_\infty \subseteq c_A$ and hence A is not coregular. Thorpe using [3] further generalized Kuttner's work by showing that any locally convex FK *space* which contains the sequences which are strongly p-Cesàro summable to 0 for any p, $0 < p < 1$ must also contain ℓ_∞ setwise. It is clear that both of these results can be extended to include the index $p = 0$ using corollary 5.3.1 and that theorem 5.3.2 can be regarded as an extention of Kuttner's theorem.

We now turn our attention to the matrices which map bsc into c. Recall that theorem 5.2.2 demonstrates that bsc$= w_p \cap \ell_\infty$ for any p, $0 \leq p < \infty$. Maddox [28] established a chacterization of such matrices starting from the viewpoint that bsc$= w_1 \cap \ell_\infty$. Sember and Freedman sharpened and gave a new proof of the result in later [14] using a property of the strongly 1-Cesáro convergent sequences of 0's and 1's.

In this chapter we will establish a result similar to the results mentioned above starting from the viewpoint that bsc$= w_0 \cap \ell_\infty$ using the decomposition theorem of the preceeding section. The fundamental observation in this demonstration is that such matrices must map statistically null sequences of 0's and 1's into null sequences. We begin with a definition.

Definition 5.3.1. Let $s = \{s_i\}$ be a strictly increasing sequence of integers with $1 \leq s_1$. We say that $s \in \mathbb{S}$ if $\lim_m (s_{2m})^{-1} \sum_{\ell=1}^{m} (s_{2\ell} - s_{2\ell-1}) = 0$.

Loosely speaking $\mathbb{S}$ is the set of supports of the divergent statistically null sequences; this connection will be made precise in the next lemma. Now we need to develop some more notation. We let θ denote a divergent sequence of 0's and 1's and s denote a strictly increasing sequence of natural numbers for the remainder of this section. Given a $\theta = \{\theta_k\}$ we define $s^{(\theta)}$ by the relations:

if $s^{(\theta)}_{2\ell-1} \leq k < s^{(\theta)}_{2\ell}$, then $\theta_k = 1$, and

if $s^{(\theta)}_{2\ell} \leq k < s^{(\theta)}_{2\ell+1}$, or $k < s_1$, then $\theta_k = 0$.

for $\ell = 1, 2, 3, \ldots$. Similarlly, for a given $s = \{s_k\}$ we define $\theta^{(s)}$ by

$\theta^{(s)}_k = 1$ if $s_{2\ell-1} \leq k < s_{2\ell}$, and

$\theta^{(s)}_k = 0$ if $s_{2\ell} \leq k < s_{2\ell+1}$, or $k < s_1$.

We can now state the connection between $\mathbb{S}$ and the statistically null sequences [8].

Lemma 5.3.2. Let s be a strictly increasing sequence of natural numbers. Then $\theta^{(s)}$ is statistically null if and only if $s \in \mathbb{S}$.

Proof. Let θ be statistically null and set $s = s^{(\theta)}$. Since θ is statistically null we have that

$$\lim_n n^{-1}|\{k \leq n : \theta_k = 1\}| = \lim_n n^{-1} \sum_{k=1}^{n} \theta_k = 0$$

Now by construction, $(s_{2m})^{-1} \sum_{\ell=1}^{m} (s_{2\ell} - s_{2\ell-1}) = (s_{2m})^{-1} \sum_{\ell=1}^{s_{2m}} \theta_\ell$, and since $\{(s_{2m})^{-1} \sum_{\ell=1}^{s_{2m}} \theta_\ell\}$ is a subsequence of a null sequence we have that

$$\lim_p (s_{2p})^{-1} \sum_{\ell=1}^{s_{2p}} (s_{2\ell} - s_{2\ell-1}) = 0, \ i.e.\ s \in \mathbb{S}.$$

Conversely, suppose $s \in \mathbb{S}$ and let $\theta = \theta^{(s)}$. Now we compute a couple of estimates for $n^{-1}|\{k \leq n : \theta_k = 1\}|$. First for a given n define $p(n)$ by requiring the relation $s_{2p(n)} \leq n < s_{2p(n)+2}$ to hold. We consider two cases:

if $s_{2p(n)} \leq n < s_{2p(n)+1}$ we have that

$$|\{k \leq n : \theta_k = 1\}| = \sum_{k=1}^{s_{2p(n)}} \theta_k \ = \ \sum_{\ell=1}^{p(n)} (s_{2\ell} - s_{2\ell-1}),$$

hence

$$(1/n)|\{k \leq n : \theta_k = 1\}| \leq (1/s_{2p(n)}) \sum_{\ell=1}^{p(n)} (s_{2\ell} - s_{2\ell-1}),$$

by the construction of θ and since $n^{-1} \leq (s_{2p(n)})^{-1}$.

In the second case we suppose that $s_{2p(n)+1} \leq n < s_{2p(n)+2}$. By the construction of θ we have that

$$(1/n)|\{k \leq n : \theta_k = 1\}| \leq (1/s_{2p(n)}) \sum_{\ell=1}^{p(n)} (s_{2\ell} - s_{2\ell-1}) \ + \ (1/n)(n - s_{2p(n)+1})$$

$$\leq (1/s_{2p(n)}) \sum_{\ell=1}^{p(n)} (s_{2\ell} - s_{2\ell-1}) \ + \ (1 - (s_{2p(n)+1}/s_{2p(n)+2})).$$

Now note that since $s \in \mathbb{S}$, we have that $\lim_p (1 - (s_{2p(n)+1}/s_{2p(n)+2}) = 0$. Consequently, given an $\epsilon > 0$ there is a p_ϵ such that for all $p \geq p_\epsilon$ we have that

$$\max \left\{ (1/s_{2\ell}) \sum_{\ell=1}^{p} (s_{2\ell} - s_{2\ell-1}) \ , \ (1 - (s_{2p(n)+1}/s_{2p(n)+2})) \right\} < \epsilon/2.$$

It follows that if $n \geq N_\epsilon = s_{2p_{\epsilon+1}}$, then $(1/n)|\{k \leq n : \theta_k = 1\}| < \epsilon$. Since ϵ was arbitrary, θ is statistically null.

We are ready to establish a characterization of the matrices which map the bounded statistically null sequences into null sequences.

Theorem 5.3.3. Let $A = (a_{n,k})$ be a matrix. The matrix A maps bounded statistically null sequences into null sequences if and only if A maps null sequences into null sequences and

$$\lim_n \sum_{\ell=1}^{\infty} \sum_{k=s_{2\ell-1}}^{s_{2\ell}} | a_{n,k} |= 0 \qquad (5.3.1)$$

for every $s \in \mathbb{S}$.

Proof. Necessity. Since null sequences are bounded and statistically null, it is clearly necessary for A to take null sequences into null sequences.

Now suppose that $\lim_n \sum_{\ell=1}^{\infty} \sum_{k=s_{2\ell-1}}^{s_{2\ell}} | a_{n,k} | \neq 0$ for some $s \in \mathbb{S}$, hence there is a $\delta > 0$ and an increasing sequence of natural numbers $n_1 < n_2 < \ldots$ such that

$$\sum_{\ell=1}^{\infty} \sum_{k=s_{2\ell-1}}^{s_{2\ell}} | a_{n_j,k} | \geq 2\delta$$

for every $j \in \mathbb{N}$. Using the fact that A maps c_0 into c_0 and a sliding hump construction, it is possible to find two sequences of natural numbers $\{y_j\}$ and $\{\beta_j\}$

such that $\gamma_j < \beta_j < \gamma_{j+1}$ for all $j \in \mathbb{N}$ and
$\sum_{\ell=1}^{\gamma_j} \sum_{k=s_{2\ell-1}}^{s_{2\ell}} | a_{n_j,k} |< \delta/2, \qquad \sum_{\ell=\beta_j+1}^{\infty} \sum_{k=s_{2\ell-1}}^{s_{2\ell}} | a_{n_j,k} |< \delta/2,$

and

$$\sum_{\ell=\gamma_j}^{\beta_j} \sum_{k=s_{2\ell-1}}^{s_{2\ell}} | a_{n_j,k} |> \delta.$$

Define a sequence $\{z_k\}$ by $(z_k)(a_{n_j,k}) =| a_{n_j,k} |$ if $s_{2\ell} \leq k < s_{2\ell+1}$ and $\gamma_j < \ell < \beta_j$ and $z_k = 0$ elsewhere.

Observe that by construction, $|(Az)_{n_j}| > \delta$ and hence $\lim_n |(Az)_{n_j}| \neq 0$. Also note that z has been constructed in such a fashion that if $\theta_k^{(s)} = 0$ then $z_k = 0$, i.e. $\text{supp}(z) \subseteq \text{supp}(\theta^{(s)})$ and hence z is statistically null.

This contradicts the hypothesis that A takes statistically null sequences into null sequences.

Sufficiency. Suppose that A maps null sequences into null sequences and satisfies (5.3.1). Let x be a bounded statistically null sequence and apply the decomposition theorem to write $x = y + z$ where y is a null sequences and
$\lim_n n^{-1}|\{k \leq n : z_k \neq 0\}| = 0$. We assume without loss of generality that $\|x\| \leq 1$ and hence $\|z\| \leq 1$.

Now recall from the proof of the decomposition theorem that any sequence whose support is contained in the support of z is also statistically null. We now claim that Az is a null sequence.

First note that Az exists this follows from the hypothesis that A maps c_0 into c_0 and that z is bounded. Now we define a sequence θ by $\theta_k = 1$ if $z_k \neq 0$ and 0 otherwise. As remarked above θ is also statistically null $| z_k |\leq \theta_k \leq 1$ for all $k \in \mathbb{N}$. It follows that

$$\begin{aligned}
|(Az)_n| &= \sum_{k=1}^{\infty} a_{n,k} z_k \\
&\leq \sum_{k=1}^{\infty} | a_{n,k} z_k | \\
&\leq \sum_{k=1}^{\infty} | a_{n,k} | \theta_k \\
&= \sum_{\ell=1}^{\infty} \sum_{k=s_{2\ell-1}}^{s_{2\ell}} | a_{n,k} |
\end{aligned}$$

where $s = s^{(\theta)}$. Since θ is statistically null $s^{(\theta)} \in \mathbb{S}$ and by (5.3.1) $\lim_n (Az)_n = 0$.

Now since $y \in c_0$, we have that $Ay \in c_0$ and consequently $Ax = Ay + Az \in c_0$, and hence the theorem.

It is worth noting that condition (5.3.1) in the above theorem can be replaced by the condition

$$\lim_n \sum_{\ell=1}^{\infty} \sum_{k=s_{2\ell-1}}^{s_{2\ell}} a_{n,k} = 0.$$

This follows from a result of Sember and Freedman (Proposition 6, [14]), which they in turn use to establish their characterization of the matrices which map the strongly 1-Cesàro summable sequences into the convergent sequences.

The decomposition theorem can be used to obtain the following consistency result [8].

Theorem 5.3.4. Let A be a regular matrix which maps statistically null sequences into null sequences. If x is a bounded sequences which is statistically convergent to L, then x is A-summable to L.

Maddox has used a similar version of Theorem 5.3.3 to show that the Borel matrix does not map the bounded strong p-Cesàro sequences and hence the bounded statistically convergent sequences into the convergent sequences [28]. Theorem 5.3.3 can be also used to give sufficient conditions for a subclass of the Nörlund and Nörlund type summability methods to be stronger than statistical convergence on the bounded sequences. Firs we give a definition.

Definition 5.3.2. Let $p = \{p_k\}$ be nonnegative sequence of real numbers with $p_1 > 0$ and set $p_n = \sum_{k=1}^{n} p_k$. The sequence p generates the Noriund mean N_p and Noriund type mean R_p as follows:

(a) $N_p = (q_{n,k})$ where $q_{n,k} = p_{n-k}/p_n$ if $k \leq n$ and $q_{n,k} = 0$ otherwise,

(b) $R_p = (r_{n,k})$ where $r_{n,k} = p_k/p_n$ if $k \leq n$ and $r_{n,k} = 0$ otherwise.

Observe that the Cesáro matrix is an example of both a Nörlund and a Nörlund type mean. It is generated the sequence $p = \{p_k\}$ where $p_k = 1$ for all $k \in \mathbb{N}$.

Lemma 5.3.3. Let $p = \{p_k\}$ generate the Nöriund and Nörlund type means N_p and R_p respectively. If p and $\{n/p_n\}$ are bounded sequences, then N_p and R_p satisfy condition (5.3.1) of Theorem 5.3.3.

Proof. Suppose M and K are positive numbers such that $p_n < M$ and $n/p_n < K$ for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and $s \in \mathbb{S}$ and suppose that $s_{2p} < n \leq s_{2p-2}$.

First consider the Noriund type means. Let $R_p = (r_{n,k})$ and observe that if $s_{2p+1} < n \leq s_{2p+2}$ then,

$$\sum_{\ell=1}^{\infty} \sum_{k=s_{2\ell-1}}^{s_{2\ell}} r_{n,k} = (1/p_n)\Big(\sum_{\ell=1}^{p} \sum_{k=s_{2\ell-1}}^{s_{2\ell}} p_k + \sum_{k=s_{2\ell+1}+1}^{n} p_k\Big)$$
$$\leq M(n/p_n)\big((1/n)\sum_{\ell=1}^{p}(s_{2\ell} - s_{2\ell-1}) + (1/n)(n - s_{2\ell-1})\big)$$
$$\leq MK\Big((1/s_{2p})\sum_{\ell=1}^{p}(s_{2\ell} - s_{2\ell-1}) + (1 - s_{2\ell+1}/s_{2\ell+2})\Big)$$
$$= A$$

A similar computation shows that if $s_{2p} < n \leq s_{2p+1}$ then,

$$\sum_{\ell=1}^{\infty} \sum_{k=s_{2\ell-1}}^{s_{2\ell}} r_{n,k} = MK(1/s_{2p})\sum_{\ell=1}^{p}(s_{2\ell} - s_{2\ell-1}) = B.$$

If $\epsilon > 0$ is given the proof of lemma 5.3.2 and the definition of $\mathbb{S}$ yield the existence of P such that $\max\{A, B\} < \epsilon$ for all $p \geq P$. If we set $N_\epsilon = s_{2p+1}$, it follows that

$$\sum_{\ell=1}^{\infty} \sum_{k=s_{2\ell-1}}^{s_{2\ell}} r_{n,k} \leq \epsilon$$

for all $n \geq N_\epsilon$. Hence the claim is established for R_p.

The proof for N_p follows in exactly the same fashion. In fact there is only one change in notation in the above computations, hence we omit it.

The necessary and sufficient conditions on p for N_p and R_p to be regular are well known and easily obtained from the Silverman-Toeplitz theorem. N_p is a regular if and only if $\lim_n(p_n/P_n) = 0$ and R_p is regular if and only if $\lim_n P_n = 0$. The proof of the following theorem now follows immediately from the lemma.

Theorem 5.3.5. Let $p = \{p_k\}$ be nonnegative sequence with $p_1 > 0$ and let $p_n = \sum_{k=1}^{n} p_k$. If the sequences $\{p_k\}$ and $\{n/p_n\}$ are bounded, then the Nörlund and Nörlund type means generated by P are both regular methods which are stronger than and consistent with statistical convergence on the bounded sequences.

Proof. The follows immadiately from noting that if $\{n/p_n\}$ is bounded, then $p_n \to \infty$ and $n \to \infty$.

CHAPTER 6

STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES

6.1. Introduction

The idea of statistical convergence for double sequences $x = (x_{jk})$ was introduced by Mursaleen and Edely [30] which is presented in this chapter.

By the convergence of a double sequence we mean the *convergence in Pringshe -im's sense* [35]. A double sequence $x = (x_{jk})_{j,k=0}^{\infty}$ is said to be convergent in the Pringsheim's sense if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\mid x_{jk} - L \mid < \epsilon$ whenever $j, k \geq N$. L is called the Pringsheim limit of x.

A double sequence $x = (x_{jk})$ is said to be *Cauchy sequence* if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\mid x_{pq} - x_{jk} \mid < \epsilon$ for all $p \geq j \geq N,\ q \geq k \geq N$.

A double sequence x is *bounded* if there exists a positive number M such that $\mid x_{jk} \mid < M$ for all j and k, i.e., if

$$\|x\|_{(\infty,2)} = \sup_{j,k} \mid x_{jk} \mid < \infty \tag{6.1.1}$$

We will denote the set of all bounded double sequences by ℓ_{∞}^2. Note that in contrast to the case for single sequences, a convergent double sequence need not be bounded.

6.2. Statistical convergence of double sequences

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K(n, m)$ be the numbers of (i, j) in K such that $i \leq n$ and $j \leq m$. Then the two-dimensional analogue of natural density can be defined as follows.

The lower asymptotic density of a set $K \subseteq \mathbb{N} \times \mathbb{N}$ is defined as

$$\delta_2(K) = \lim_{n,m} inf \frac{K(n,m)}{nm}.$$

In case the sequence $(K(n, m)/nm)$ has a limit in Pringsheim's sense then we say that K has a double natural density and is defined as

$$\lim_{n,m} \frac{K(n,m)}{nm} = \delta_2(K).$$

For example, let $K = \{(i^2, j^2) : i, j \in \mathbb{N}\}$. Then

$$\delta_2(K) = \lim_{n,m} \frac{K(n,m)}{nm} \leq \lim_{n,m} \frac{\sqrt{n}\,\sqrt{m}}{nm} = 0.$$

i.e., the set K has double natural density zero, while the set $\{(i, 2j) : i, j \in \mathbb{N}\}$ has double natural density 1/2.

Note that, if we set $n = m$, we have a two-dimensional natural density considered by Christopher [6].

We define the statistical analogue for double sequences $x = (x_{jk})$ as follows.

Definition 6.2.1. A real double sequence $x = (x_{jk})$ is said to be statistically convergent to the number ℓ if for each $\epsilon > 0$, the set

$$\{(i,j),\ j \leq n \text{ and } k \leq m : |x_{jk} - l| \geq \epsilon\}$$

has double natural density zero. In this case we write $st_2\text{-}\lim_{j,k} x_{jk} = \ell$ and we denote the set of all statistically convergent double sequences by st_2.

Remark 6.2.1. (a) If x is a convergent double sequence then it is also statistically convergent to the same number. Since there are only a finite number of bounded (unbounded) rows and/or columns,

$$K(n,m) \leq s_1 n + s_2 m,$$

where s_1 and s_2 are finite numbers, which we can conclude that x is statistically convergent.

(b) If x is statistically convergent to the number l ,then l is determined uniquely.

(c) If x is statistically convergent, then x need not be convergent. Also it is not necessarily bounded. For example, let $x = (x_{jk})$ be defined as

$$x_{jk} = \begin{cases} jk, & \text{if } j \text{ and } k \text{ are squares} \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to see that $st_2\text{-}\lim x_{jk} = 1$, since the cardinality of the set $\{(j,k) : |x_{jk} - 1| \geq \epsilon\} \leq \sqrt{j}\sqrt{k}$ for every $\epsilon > 0$. But x is neither convergent nor bounded.

We prove some analogues for double sequences. For single sequences such results have been proved by Salat [36].

Theorem 6.2.1. A real double sequence $x = (x_{jk})$ is statistically convergent to a number ℓ if and only if there exists a subset $K = \{(j,k)\} \subseteq \mathbb{N} \times \mathbb{N},\ j,k = 1,2,...,$ such that $\delta_2(K) = 1$ and

$$\lim_{\substack{j,k\to\infty \\ (j,k)\in K}} x_{jk} = \ell$$

. **Proof.** Let x be statistically convergent to ℓ. Put

$$K_r = \{(j,k) \in \mathbb{N} \times \mathbb{N} :\mid x_{jk} - \ell \mid \geq \tfrac{1}{r}\}$$

and

$$M_r = \{(j,k) \in \mathbb{N} \times \mathbb{N} : |x_{jk} - \ell| < \frac{1}{r}\} \quad (r = 1,2,3,...).$$

Then $\delta_2(K_r) = 0$ and

$$(1)\quad M_1 \supset M_2 \supset .... \supset M_i \supset M_{i+1} \supset ....$$

and

$$(2)\quad \delta_2(M_r) = 1, \quad r = 1,2,3,...$$

Now we have to show that for $(j,k) \in M_r$, (x_{jk}) is convergent to ℓ. Suppose that (x_{jk}) is not convergent to ℓ. Therefore there is $\epsilon > 0$ such that

$$\mid x_{jk} - \ell \mid \geq \epsilon \text{ for infinitely many terms.}$$

Let

$$M_\epsilon = \{(j,k) :\mid x_{jk} - \ell \mid < \epsilon\} \text{ and } \epsilon > \tfrac{1}{r} \ (r = 1,2,...).$$

Then

$$(3)\quad \delta_2(M_\epsilon) = 0,$$

and by (1), $M_r \subset M_\epsilon$. Hence $\delta_2(M_r) = 0$ which contradicts (2). Therefore (x_{jk}) is convergent to ℓ.

Conversely, suppose that there exists a subset $K = \{(j,k)\} \subseteq \mathbb{N} \times \mathbb{N}$ such that $\delta_2(K) = 1$ and $\lim_{j,k} x_{jk} = \ell$ i.e., there exists $N \in \mathbb{N}$ such that for every $\epsilon > 0$,

$$\mid x_{jk} - l \mid < \epsilon, \quad \forall\ j,k \geq N.$$

Now

$$K_\epsilon = \{(j,k) : \mid x_{jk} - \ell \mid \geq \epsilon\} \subseteq \mathbb{N} \times \mathbb{N} - \{(j_{N+1}, k_{N+1}), (j_{N+2}, k_{N+2}), \ldots\}.$$

Therefore

$$\delta_2(K_\epsilon) \leq 1 - 1 = 0.$$

Hence x is statistically convergent to ℓ.

Remark 6.2.2. If $st\text{-}\lim_{jk} x_{jk} = \ell$, then there exists a sequence $y = (y_{jk})$ such that $\lim_{j,k} y_{jk} = \ell$ and $\delta_2\{(j,k) : x_{jk} = y_{jk}\} = 1$, i.e.,

$$x_{jk} = y_{jk} \quad \text{for almost all } j,k \text{ (for short } a.\ a.\ j,k).$$

Theorem 6.2.2. The set $st_2 \cap \ell_\infty^2$ is a closed linear subspace of the normed linear space ℓ_∞^2.

Proof. Let $x^{(nm)} = (x_{jk}^{(nm)}) \in st_2 \cap \ell_\infty^2$ and $x^{(nm)} \to x \in \ell_\infty^2$. Since $x^{(nm)} \in st_2 \cap \ell_\infty^2$, there exist real numbers a_{nm} such that

$$st_2\text{-}\lim_{j,k} x_{jk}^{(nm)} = a_{nm} \quad (n, m = 1, 2, \ldots).$$

As $x^{(nm)} \to x$, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\mid x^{(pq)} - x^{(nm)} \mid < \epsilon/3 \tag{6.2.1}$$

for every $p \geq n \geq N, q \geq m \geq N$, where $\mid . \mid$ denotes the norm in a linear space.

By Theorem 6.2.1, there exist subsets K_1 and K_2 of $\mathbb{N} \times \mathbb{N}$ with $\delta_2(K_1) = \delta_2(K_2) = 1$ and

(1) $\lim\limits_{\substack{j,k\to\infty \\ (j,k)\in K_1}} x_{jk}^{(nm)} = a_{nm}$

(2) $\lim\limits_{\substack{j,k\to\infty \\ (j,k)\in K_2}} x_{jk}^{(pq)} = a_{pq}$.

Now the set $K_1 \cap K_2$ is infinite since $\delta_2(K_1 \cap K_2) = 1$.

Choose $(k_1, k_2) \in K_1 \cap K_2$. We have from (1) and (2) that

$$\mid x_{k_1,k_2}^{(pq)} - a_{pq} \mid < \epsilon/3 \tag{6.2.2}$$

and

$$\mid x_{k_1,k_2}^{(nm)} - a_{nm} \mid < \epsilon/3. \tag{6.2.3}$$

Therefore for each $p \geq n \geq N$ and $q \geq m \geq N$ we have from (6.2.1) – (6.2.3),

$$\begin{aligned} | a_{pq} - a_{nm} | &\leq | a_{pq} - x^{pq}_{k_1,k_2} | + | x^{pq}_{k_1,k_2} - x^{nm}_{k_1,k_2} | + | x^{nm}_{k_1,k_2} - a_{nm} | \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon. \end{aligned}$$

That is, the sequence (a_{nm}) is a Cauchy sequence and hence convergent. Let

$$\lim_{n,m} a_{nm} = a. \tag{6.2.4}$$

We need to show that x is statistically convergent to a. Since $x^{(nm)}$ is convergent to x, for every $\epsilon > 0$ there is $N_1(\epsilon)$ such that for $j, k \geq N_1(\epsilon)$,

$$| x^{(nm)}_{jk} - x_{jk} | < \epsilon/3.$$

Also from (6.2.4) we have for every $\epsilon > 0$ there is $N_2(\epsilon)$ such that for all $j, k \geq N_2(\epsilon)$,

$$| a_{jk} - a | < \epsilon/3.$$

Again, since $x^{(nm)}$ is statistically convergent to a_{nm}, there exists a set $K = \{(j,k)\} \subseteq \mathbb{N} \times \mathbb{N}$ such that $\delta_2(K) = 1$ and for every $\epsilon > 0$ there is $N_3(\epsilon)$ such that for all $j, k \geq N_3(\epsilon)$, $(j,k) \in K$,

$$| x^{(nm)}_{jk} - a_{nm} | < \epsilon/3.$$

Let $\max\{N_1(\epsilon), N_2(\epsilon), N_3(\epsilon)\} = N_4(\epsilon)$. Then for a given $\epsilon > 0$ and for all $j, k \geq N_4(\epsilon)$, $(j,k) \in K$,

$$| x_{jk} - a | \leq | x_{jk} - x^{(nm)}_{jk} | + | x^{(nm)}_{jk} - a_{jk} | + | a_{jk} - a | < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Therefore x is statistically convergent to a, i.e., $x \in st_2 \cap \ell^2_\infty$.

Hence $st_2 \cap \ell^2_\infty$ is a closed linear subspace of ℓ^2_∞.

Theorem 6.2.3. The set $st_2 \cap \ell^2_\infty$ is nowhere dense in ℓ^2_∞.

Proof. Since every closed linear subspace of an arbitrary linear normed space S different from S is a nowhere dense set in S ([34]), by Theorem 6.2.2 we need only to show that $st_2 \cap \ell^2_\infty \neq \ell^2_\infty$.

Let the sequence $x = (x_{jk})$ be defined by

$$x_{jk} = \begin{cases} 1, & \text{if } j \text{ and } k \text{ are even} \\ \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that x is not statistically convergent but x is bounded. Hence $st_2 \cap \ell^2_\infty \neq \ell^2_\infty$.

6.3. Statistically Cauchy double sequences

In [16] Fridy has defined the concept of statistically Cauchy single sequences. In this section we define statistically Cauchy double sequences and prove some analogues.

Definition 6.3.1. A real double sequence $x = (x_{jk})$ is said to be statistically Cauchy if for every $\epsilon > 0$ there exist $N = N(\epsilon)$ and $M = M(\epsilon)$ such that for all $j, p \geq N, k, q \geq M$, the set

$$\{(j,k),\ j \leq n,\ k \leq m : \mid x_{jk} - x_{pq} \mid \geq \epsilon\}$$

has natural density zero.

Theorem 6.3.1. A real double sequence $x = (x_{jk})$ is statistically convergent if and only if x is statistically Cauchy.

Proof. Let x be statistically convergent to a number ℓ. Then for every $\epsilon > 0$, the set

$$\{(j,k),\ j \leq n,\ k \leq m : \mid x_{jk} - \ell \mid \geq \epsilon\}$$

has natural density zero. Choose two numbers N and M such that $\mid x_{NM} - \ell \mid \geq \epsilon$. Now let

$$A_\epsilon = \{(j,k),\ j \leq n,\ k \leq m : \mid x_{jk} - x_{NM} \mid \geq \epsilon\},$$

$$B_\epsilon = \{(j,k),\ j \leq n,\ k \leq m : \mid x_{jk} - \ell \mid \geq \epsilon\},$$

$$C_\epsilon = \{(j,k),\ j = N \leq n,\ k = M \leq m : \mid x_{NM} - \ell \mid \geq \epsilon\}.$$

Then $A_\epsilon \subseteq B_\epsilon \cup C_\epsilon$ and therefore $\delta_2(A_\epsilon) \leq \delta_2(B_\epsilon) + \delta_2(C_\epsilon) = 0$. Hence x is statistically Cauchy.

Conversely, let x be statistically Cauchy but not statistically convergent. Then there exist N and M such that the set A_ϵ has natural density zero. Hence the set

$$E_\epsilon = \{(j,k),\ j \leq n,\ k \leq m : \mid x_{jk} - x_{NM} \mid < \epsilon\}$$

has natural density 1. In particular, we can write

$$\mid x_{jk} - x_{NM} \mid \leq 2 \mid x_{jk} - \ell \mid < \epsilon \tag{6.3.1}$$

if $\mid x_{jk} - \ell \mid < \epsilon/2$. Since x is not statistically convergent, the set B_ϵ has natural density 1, i.e., the set

$$\{(j,k),\ j \leq n,\ k \leq m : \mid x_{jk} - \ell \mid < \epsilon\}$$

has natural density 0. Therefore by (6.3.1) the set

$$\{(j,k),\ j \leq n,\ k \leq m : \mid x_{jk} - x_{NM} \mid < \epsilon\}$$

has natural density 0, i.e., the set A_ϵ has natural density 1 which is a contradiction. Hence x is statistically convergent.

From Theorems 6.2.1 and 6.3.1 we can state the following for double sequences analogous to the result of Fridy [16].

Theorem 6.3.2. The following statements are equivalent:

(a) x is statistically convergent to ℓ

(b) x is statistically Cauchy

(c) there exists a subsequence y of x such that $\lim_{jk} y_{jk} = \ell$.

6.4. Relation between statistical convergence and strongly Cesàro Summable Sequences

The following definitions of Cesàro summable double sequences is taken from [29].

Definition 6.4.1. Let $x = (x_{jk})$ be a double sequence. It is said to be Cesàro summable to ℓ if

$$\lim_{n,m} \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} = \ell.$$

We denote the space of all Cesàro summable double sequences by $(C,1,1)$.

Similarly we can define the following as in case of single sequences.

Definition 6.4.2. Let $x = (x_{jk})$ be a double sequence and p be a positive real number. Then the double sequence x is said to be strongly p-Cesàro summable to ℓ if

$$\lim_{n,m} \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} \mid x_{jk} - \ell \mid^{p} = 0.$$

We denote the space of all strongly p-Cesàro summable double sequences by w_p^2.

Remark 6.4.1. (i) If $0 < p \leq q < \infty$, then $w_q^2 \subseteq w_p^2$ (by Hölder's inequality) and

$$w_p^2 \cap \ell_\infty^2 = w_1^2 \cap \ell_\infty^2 \subseteq (C,1,1) \cap \ell_\infty^2.$$

(ii) If x is convergent but unbounded then x is statistically convergent but x need not be Cesàro nor strongly Cesàro.

Example 6.4.1. Let $x = (x_{jk})$ be defined as

$$x_{jk} = \begin{cases} k, & j = 1, \text{ for all } k \\ j, & k = 1, \text{ for all } j \\ 0, & \text{otherwise.} \end{cases}$$

Then $\lim_{j,k} x_{jk} = 0$ but

$$\lim_{n,m} \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} = \lim_{n,m} \frac{1}{nm} \frac{1}{2}(m^2 + n^2 + m + n - 2),$$

which does not tend to a finite limit. Hence x is not Cesàro. Also x is not strongly Cesàro but

$$\lim_{n,m} \frac{1}{nm} \big|\{(j,k) : |x_{jk} - 0| \geq \epsilon\}\big| = \lim_{n,m} \frac{m+n-1}{nm} = 0,$$

i.e., x is statistically convergent to 0.

(iii) If x is a bounded convergent double sequence then it is also $(C,1,1)$, w_p^2 and st_2.

The following result is analogue of Theorem 2.1 due to Connor [9].

Theorem 6.4.1. Let $x = (x_{jk})$ be a double sequence and p be a positive real number. Then

(a) x is statistically convergent to ℓ if it is strongly p-Cesàro summable to ℓ

(b) $w_p^2 \cap \ell_\infty^2 = st_2 \cap \ell_\infty^2$.

Proof. (a) Let $K_\epsilon(p) = \{(j,k),\ j \leq n,\ k \leq m : \mid x_{jk} - \ell \mid^p \geq \epsilon\}$. Now since x is strongly p-Cesàro summable to ℓ,

$$\begin{aligned} 0 \leftarrow & \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} \mid x_{jk} - \ell \mid^p \\ = & \frac{1}{nm} \Big\{ \sum_{(j,k) \in K_\epsilon(p)} \mid x_{jk} - \ell \mid^p + \sum_{(j,k) \notin K_\epsilon(p)} \mid x_{jk} - \ell \mid^p \Big\} \\ \geq & \frac{1}{nm} \Big| \Big\{ (j,k),\ j \leq n,\ k \leq m : \mid x_{jk} - \ell \mid^p \geq \epsilon \Big\} \Big| \epsilon. \end{aligned}$$

Hence x is statistically convergent to ℓ.

(b) Let $I_\epsilon(p) = \{(j,k),\ j \leq n,\ k \leq m : \mid x_{jk} - \ell \mid \geq (\epsilon/2)^{1/p}\}$ and $M = \|x\|_{(\infty,2)} + \mid \ell \mid$, where $\|x\|_{(\infty,2)}$ is the sup-norm for bounded double sequences $x = (x_{jk})$ given by (1.1).

Since x is a bounded statistically convergent, we can choose $N = N(\epsilon)$ such that for all $n, m \geq N$,

$$\frac{1}{nm}\left|\left\{(j,k),\ j \leq n,\ k \leq m : \mid x_{jk} - \ell \mid \geq \left(\frac{\epsilon}{2}\right)^{1/p}\right\}\right| < \frac{\epsilon}{2M^p}.$$

Now for all $n, m \geq N$ we have

$$\frac{1}{nm}\sum_{j=1}^{n}\sum_{k=1}^{m} \mid x_{jk} - \ell \mid^p = \frac{1}{nm}\left\{ \sum_{(j,k)\in I_\epsilon(p)} \mid x_{jk} - \ell \mid^p + \sum_{(j,k)\notin I_\epsilon(p)} \mid x_{jk} - \ell \mid^p \right\}$$

$$< \frac{1}{nm} nm \frac{\epsilon}{2M^p} M^p + \frac{1}{nm} nm \frac{\epsilon}{2} = \epsilon.$$

Hence x is strongly p-Cesàro summable to ℓ.

Remark 6.4.2. Note that if a bounded sequence x is statistically convergent then it is also $(C,1,1)$ summable but not conversely.

Example 6.4.2. Let $x = (x_{jk})$ be defined by

$$x_{jk} = (-1)^j, \quad \forall\ k,$$

then

$$\lim_{n,m} \frac{1}{nm}\sum_{j=1}^{n}\sum_{k=1}^{m} x_{jk} = 0,$$

but obviously x is not statistically convergent.

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